Dirac's Observables for the Higgs Model: I) the Abelian Case.

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Abstract

We search a canonical basis of Dirac's observables for the classical Abelian Higgs model with fermions in the case of a trivial U(1) principal bundle. The study of the Gauss law first class constraint shows that the model has two disjoint sectors of solutions associated with two physically different phases. In the electromagnetic phase, the electromagnetic field remains massless: after the determination of the Dirac's observables we get that both the reduced physical Hamiltonian and Lagrangian are nonlocal. In the Higgs phase, the electromagnetic field becomes massive and in terms of Dirac's observables we get a local, but nonanalytic in the electric charge (or equivalently in the sum of the electromagnetic mass and of the residual Higgs field), physical Hamiltonian; however the associated Lagrangian is nonlocal. Some comments on the R-gauge-fixing, the possible elimination of the residual Higgs field and on the Nielsen-Olesen vortex solution close the paper.

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I. INTRODUCTION

After having found a symplectic basis of Dirac's observables for the classical Yang-Mills theory with Grassmann-valued fermions Ref. [1] in the case of a trivial principal bundle over Minkowski spacetime and in suitable function spaces where the Gribov ambiguity is absent, the next step in the program [2] of reformulating particle physics in terms of Dirac's observables is the study of the Higgs model. This model is needed to generate the spontaneous symmetry breaking used in the $SU(2) \times U(1)$ electroweak standard model to give mass to the vector gauge bosons and, through the Yukawa couplings, to the fermions. Here, we shall preliminary study the classical Abelian Higgs model with fermions [trivial U(1) principal bundle] to disentangle the basic implications of the Higgs mechanism from the complications of the $SU(2) \times U(1)$ model.

The Abelian Higgs model is described by the following Lagrangian density [$\lambda > 0$, $\phi_o > 0$]

$$
\begin{aligned}
\mathcal{L}(x) &= -\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x) + [D^{(A)}_\mu \phi(x)]^* \, D^{(A)\mu} \phi(x) - V(\phi) + \\
&\quad + \frac{1}{2} \bar{\psi}(x) [\gamma^\mu (i\partial_\mu + eA_\mu(x)) - \, (i\overleftarrow{\partial_\mu - e} A_\mu(x)) \, \gamma^\mu] \psi(x) - m\bar{\psi}(x)\psi(x) \\
V(\phi) &= \lambda [\phi^*(x)\phi(x) - \phi_o^2]^2 = \mu^2 \phi^*(x)\phi(x) + \lambda [\phi^*(x)\phi(x)]^2 + \lambda \phi_o^4 = \\
&= -\frac{1}{2} m_H^2 \phi^*(x)\phi(x) + \lambda [\phi^*(x)\phi(x)]^2 + \lambda \phi_o^4,
\end{aligned}
$$

$$
\mu^2 = -2\lambda\phi_o^2 < 0, \qquad m_H^2 = -2\mu^2 = 4\lambda\phi_o^2, \qquad \phi_o = \frac{m_H}{2\sqrt{\lambda}} = \sqrt{\frac{-\mu^2}{2\lambda}}, \tag{1}
$$

where $\phi(x)$, the Higgs field, is a complex scalar field [$D^{(A)}_\mu \phi(x) = (\partial_\mu - ieA_\mu(x))\phi(x)$], $\mu^2 < 0$ so that the potential $V(\phi)$ has a set of absolute minima for $\phi^*\phi = \phi_o^2$, parametrized by a phase [$\phi \mapsto e^{i\theta}\phi$ leaves $\phi^*\phi$ invariant], and with $\phi_o > 0$ an arbitrary real number [$< \phi >= \phi_o \neq 0$ at the quantum level: this is the gauge non-invariant formulation of the statement of symmetry breaking].The fermion field $\psi(x)$ is Grassmann-valued and is absent when the Abelian Higgs model is used as a relativistic generalization [3,4] of the Landau-Ginzburg treatment [5] of superconductivity with $\phi(x)$ [also called the "complex order parameter", with the ordered phase being the broken symmetry one] associated with the spin-singlet part of the nonzero vacuum expectation value of the fermion bilinear describing the Bose-Einstein condensation of the Cooper electron pairs [the attractive effects of virtual phonons being slightly higher of the Coulomb repulsion] and with the massless Goldstone boson, generated by the spontaneous symmetry breaking, reabsorbed to give mass to the photon so to obtain a finite-range electromagnetic field as required by the Meissner effect of magnetic flux exclusion [6] (physically the longitudinal degrees of freedom of the electromagnetic field couple to the plasma oscillations, i.e. to the collective density fluctuations of the electrons).

The Lagrangian density is invariant under the U(1) gauge transformations $A_\mu(x) \mapsto A_\mu(x) - \frac{1}{e}\partial_\mu\alpha(x)$, $\phi(x) \mapsto e^{-i\alpha(x)}\phi(x)$, $\psi(x) \mapsto e^{-i\alpha(x)}\psi(x)$.

We shall show that the singular Lagrangian density of Eq.(1) describes simultaneously two extremely different dynamics, since its associate Gauss law constraint (or equivalently the corresponding acceleration-independent Euler-Lagrange equation) generates two disjoint sectors of solutions and only one of them (the electromagnetic phase with massless electromagnetic fields) is analytic in the coupling constant (the electric charge). To describe these two sectors, i.e. the electromagnetic and Higgs phases respectively, we shall use different parametrizations of the Higgs fields.

II. THE ELECTROMAGNETIC PHASE

The canonical momenta associated with Eq.(1) are

$$\begin{aligned}
&\pi^o(x) = 0 \\
&\vec{\pi}(x) = \vec{E}(x) \\
&\pi(x) = -\frac{i}{2}\psi^\dagger(x) \\
&\bar{\pi}(x) = -\frac{i}{2}\gamma^o\psi(x) \\
&\pi_\phi(x) = \dot{\phi}^*(x) + ieA_o(x)\phi^*(x) \\
&\pi_{\phi*}(x) = \dot{\phi}(x) - ieA_o(x)\phi(x),
\end{aligned} \tag{2}$$

and are assumed to satisfy the Poisson brackets

$$\{A_\mu(\vec{x},x^o),\pi^\nu(\vec{y},x^o)\} = \delta^\nu_\mu \delta^3(\vec{x}-\vec{y})$$
$$\{\psi_\alpha(\vec{x},x^o),\pi_\beta(\vec{y},x^o)\} = \{\bar{\psi}_\alpha(\vec{x},x^o),\bar{\pi}_\beta(\vec{y},x^o)\} = -\delta_{\alpha\beta}\delta^3(\vec{x}-\vec{y})$$
$$\{\phi(\vec{x},x^o),\pi_\phi(\vec{y},x^o)\} = \{\phi^*(\vec{x},x^o),\pi_{\phi*}(\vec{y},x^o)\} = \delta^3(\vec{x}-\vec{y}). \tag{3}$$

By eliminating the fermionic second class constraints with the introduction of the Dirac brackets

$$\{\psi_\alpha(\vec{x},x^o),\bar{\psi}_\beta(\vec{y},x^o)\}^* = -i(\gamma^o)_{\alpha\beta}\delta^3(\vec{x}-\vec{y}) \tag{4}$$

(denoted $\{.,.\}$ in the rest of the paper for the sake of simplicity) as shown in Ref. [1], one arrives at the following Dirac Hamiltonian density [$\lambda_o(x)$ is a Dirac multiplier and an integration by parts has been done]

$$\begin{aligned}
\mathcal{H}_D(x) = &\frac{1}{2}[\vec{\pi}^2(x)+\vec{B}^2(x)] + \psi^\dagger(x)\vec{\alpha}\cdot(i\vec{\partial}+e\vec{A}(x))\psi(x) + m\bar{\psi}(x)\psi(x) + \\
&+ \pi_{\phi*}(x)\pi_\phi(x) + [(\vec{\partial}+ie\vec{A}(x))\phi^*(x)]\cdot(\vec{\partial}-ie\vec{A}(x))\phi(x) + \lambda(\phi^*(x)\phi(x)-\phi_o^2)^2 - \\
&- A_o(x)[-\vec{\partial}\cdot\vec{\pi}(x) + e\psi^\dagger(x)\psi(x) - ie(\pi_\phi(x)\phi(x)-\pi_{\phi*}(x)\phi^*(x))] + \lambda_o(x)\pi^o(x).
\end{aligned} \tag{5}$$

The constraint analysis shows that there a primary first class constraint, $\pi^o(x) \approx 0$, and a secondary first class one (the Gauss law, namely the acceleration-independent Euler-Lagrange equation of the model)

$$\Gamma(x) = -\vec{\partial}\cdot\vec{\pi}(x) + e\psi^\dagger(x)\psi(x) - ie[\pi_\phi(x)\phi(x)-\pi_{\phi*}(x)\phi^*(x)] \approx 0. \tag{6}$$

Eq.(6) is ambiguous, since it can be considered either as an elliptic equation for the electric field $\vec{\pi}$ or as an algebraic equation in the Higgs momenta: in the first case one obtains a sector of solutions corresponding to the electromagnetic phase, in the second one the sector of the Higgs phase. As a consequence, the space of solutions of the Euler-Lagrange equations is not connected being formed by two disjoint subspaces (its zeroth homotopy group is not trivial).

Since the conserved energy-momentum and angular momentum tensor densities and Poincaré generators are ["$\stackrel{\circ}{=}$" means evaluated on the extremals of the action $S=\int d^4x\mathcal{L}(x)$; $\sigma^{\mu\nu}=\frac{i}{2}[\gamma^\mu,\gamma^\nu]$, $\sigma^i=\frac{1}{2}\epsilon^{ijk}\sigma^{jk}$, $\vec{\alpha}=\gamma^o\vec{\gamma}$, $\beta=\gamma^o$]

$$
\begin{aligned}
\Theta^{\mu\nu}(x) &= F^{\mu\alpha}(x)F_\alpha{}^\nu(x) + \frac{1}{4}\eta^{\mu\nu}F^{\alpha\beta}(x)F_{\alpha\beta}(x) + \\
&+ \frac{1}{2}[\bar{\psi}(x)\gamma^\mu(i\partial^\nu + eA^\nu(x))\psi(x) - \bar{\psi}(x)\ (i\overleftarrow{\partial^\nu - e}A^\nu(x))\ \gamma^\mu\psi(x)] + \\
&+ \left(D^{(A)\mu}\phi(x)\right)^* D^{(A)\nu}\phi(x) + \left(D^{(A)\nu}\phi(x)\right)^* D^{(A)\mu}\phi(x) - \\
&- \eta^{\mu\nu}[\left(D^{(A)\alpha}\phi(x)\right)^* D^{(A)}{}_\alpha\phi(x) - V(\phi)], \\
\mathcal{M}^{\mu\alpha\beta}(x) &= x^\alpha\Theta^{\mu\beta}(x) - x^\beta\Theta^{\mu\alpha}(x) + \frac{1}{4}\bar{\psi}(x)(\gamma^\mu\sigma^{\alpha\beta} + \sigma^{\alpha\beta}\gamma^\mu)\psi(x), \\
&\quad \partial_\nu\Theta^{\nu\mu}(x) \stackrel{\circ}{=} 0, \qquad \partial_\mu\mathcal{M}^{\mu\alpha\beta}(x) \stackrel{\circ}{=} 0,
\end{aligned}
$$

$$
\begin{aligned}
P^\mu &= \int d^3x\Theta^{o\mu}(\vec{x},x^o), \\
J^{\mu\nu} &= \int d^3x\mathcal{M}^{o\mu\nu}(\vec{x},x^o),
\end{aligned}
$$

$$
\begin{aligned}
P^o &= \int d^3x\,\{\frac{1}{2}[\vec{\pi}^2(\vec{x},x^o) + \vec{B}^2(\vec{x},x^o)] + \\
&+ \pi_\phi\,(\vec{x},x^o)\pi_{\phi^*}(\vec{x},x^o) + \left(\vec{D}^{(A)}\phi(\vec{x},x^o)\right)^* \cdot \vec{D}^{(A)}\phi(\vec{x},x^o) + V(\phi) + \\
&+ \ \tfrac{1}{2}\ [\bar{\psi}(\vec{x},x^o)\gamma^o(i\partial^o + eA^o(\vec{x},x^o))\psi(\vec{x},x^o) - \bar{\psi}(\vec{x},x^o)\ (i\partial^o - \overleftarrow{eA^o}(\vec{x},x^o))\ \gamma^o\psi(\vec{x},x^o)]\} \\
P^i &= \int d^3x\,\{\left(\vec{\pi}(\vec{x},x^o)\times\vec{B}(\vec{x},x^o)\right)^i + \\
&+ \pi_\phi\,(\vec{x},x^o)D^{(A)i}\phi(\vec{x},x^o) + \left(D^{(A)i}\phi(\vec{x},x^o)\right)^*\pi_{\phi^*}(\vec{x},x^o) + \\
&+ \ \tfrac{1}{2}\ [\bar{\psi}(\vec{x},x^o)\gamma^o(i\partial^i + eA^i(\vec{x},x^o))\psi(\vec{x},x^o) - \bar{\psi}(\vec{x},x^o)\ (i\partial^i - \overleftarrow{eA^i}(\vec{x},x^o))\ \gamma^o\psi(\vec{x},x^o)]\} \\
J^i &= \frac{1}{2}\epsilon^{ijk}J^{jk} = \int d^3x\,\{[\vec{x}\times(\vec{\pi}(\vec{x},x^o)\times\vec{B}(\vec{x},x^o))]^i - \\
&- \quad [\vec{x}\times(\pi_\phi(\vec{x},x^o)\vec{D}^{(A)}\phi(\vec{x},x^o) + (\vec{D}^{(A)}\phi(\vec{x},x^o))^*\pi_{\phi^*}(\vec{x},x^o))]^i + \\
&+ \frac{1}{2}[\vec{x}\times[\bar{\psi}(\vec{x},x^o)\gamma^o(i\vec{\partial} + e\vec{A}(\vec{x},x^o))\psi(\vec{x},x^o) - \\
&- \bar{\psi}(\vec{x},x^o)\ (i\vec{\partial} - \overleftarrow{e\vec{A}}(\vec{x},x^o))\ \gamma^o\psi(\vec{x},x^o)]\,]^i\} \\
&+ \frac{1}{2}\psi^\dagger(\vec{x},x^o)\sigma^i\psi(\vec{x},x^o)\} \\
K^i &= J^{oi} = x^oP^i - \int d^3x\,x^i\Theta^{oo}(\vec{x},x^o), \qquad (7)
\end{aligned}
$$

following Dirac [7] and Ref. [1], we will assume boundary conditions $A_o(\vec{x},x^o)\rightarrow_{r\rightarrow\infty} a_o/r^{1+\epsilon}$, $\vec{A}(\vec{x},x^o)\rightarrow_{r\rightarrow\infty}\vec{a}/r^{2+\epsilon}$, $r = |\,\vec{x}\,|$, so that the Laplacian on R^3, $\triangle = -\vec{\partial}^2$, has no zero modes and the Poincaré generators are finite; this requires also the following boundary conditions on the fermion and Higgs fields: $\psi(\vec{x},x^o)\rightarrow_{r\rightarrow\infty}\chi/r^{3/2+\epsilon}+O(r^{-2})$, $\phi(\vec{x},x^o)\rightarrow_{r\rightarrow\infty} const.+\varphi/r^{2+\epsilon}+O(r^{-3})$ [the "constant" is required for the Higgs sector], $\pi_\phi(\vec{x},x^o)\rightarrow_{r\rightarrow\infty}\zeta/r^{2+\epsilon}+O(r^{-3})$. The U(1) gauge transformations are assumed to behave as $U(\vec{x},x^o)\rightarrow_{r\rightarrow\infty} const.+O(r^{-1})$, so to preserve the boundary conditions. The previous boundary conditions are adapted to the fixed x^o, not Lorentz-covariant, Hamiltonian formalism; however, they are natural in its covariantization by means of the reformulation of the theory on spacelike hypersurfaces (see Section V).

In the electromagnetic phase one obtains the following decompositions from the Hodge theorem

$$
\begin{aligned}
&\vec{A}(x)=\vec{\partial}\eta(x)+\vec{A}_\perp(x)\\
&\vec{\pi}(x)=\vec{\pi}_\perp(x)+\frac{\vec{\partial}}{\triangle}\{\Gamma(x)-e\psi^\dagger(x)\psi(x)+ie[\pi_\phi(x)\phi(x)-\pi_{\phi*}(x)\phi^*(x)]\}
\end{aligned}
$$

$$
\begin{aligned}
&\eta(\vec{x},x^o)=-\frac{1}{\triangle}\vec{\partial}\cdot\vec{A}(\vec{x},x^o)=-\int d^3y\,\vec{c}(\vec{x}-\vec{y})\cdot\vec{A}(\vec{y},x^o)\\
&\vec{c}(\vec{x})=\frac{\vec{\partial}}{\triangle}\delta^3(\vec{x}-\vec{y})=\frac{\vec{x}}{4\pi|\,\vec{x}\,|^3}\\
&A^i_\perp(x)=(\delta^{ij}+\frac{\partial^i\partial^j}{\triangle})A^j(x),\quad \pi^i_\perp(x)=(\delta^{ij}+\frac{\partial^i\partial^j}{\triangle})\pi^j(x)
\end{aligned}
$$

$$
\begin{aligned}
&\{\eta(\vec{x},x^o),\Gamma(\vec{y},x^o)\}=-\delta^3(\vec{x}-\vec{y})\\
&\{A^i_\perp(\vec{x},x^o),\pi^j_\perp(\vec{y},x^o)\}=-(\delta^{ij}+\frac{\partial^i\partial^j}{\triangle})\delta^3(\vec{x}-\vec{y}).
\end{aligned}
\tag{8}
$$

The fields $A_o(x),\pi^o(x)$ and $\eta(x),\Gamma(x)$ are pairs of conjugate gauge variables, while $\vec{A}_\perp(x),\vec{\pi}_\perp(x)$ are a canonical basis of Dirac's observables. As shown in Ref. [1], the Dirac observables for the fermion field are

$$
\begin{aligned}
&\check{\psi}(x)=e^{-ie\eta(x)}\psi(x)\\
&\check{\psi}^\dagger(x)=\psi^\dagger(x)e^{ie\eta(x)}\\
&\qquad\{\check{\psi}_\alpha(\vec{x},x^o),\check{\psi}^\dagger_\beta(\vec{y},x^o)\}=-i\delta_{\alpha\beta}\delta^3(\vec{x}-\vec{y});
\end{aligned}
\tag{9}
$$

they describe the charged fermions dressed with their Coulomb cloud.

Since

$$\{\phi(\vec{x},x^o),\Gamma(\vec{y},x^o)\} = -ie\phi(\vec{x},x^o)\delta^3(\vec{x}-\vec{y}),$$
$$\{\pi_\phi(\vec{x},x^o),\Gamma(\vec{y},x^o)\} = +ie\pi_\phi(\vec{x},x^o)\delta^3(\vec{x}-\vec{y}), \tag{10}$$

the Dirac observables for the Higgs field are

$$\begin{aligned}
\check{\phi}(x) &= e^{-ie\eta(x)}\phi(x),\\
\check{\pi}_\phi(x) &= e^{ie\eta(x)}\pi_\phi(x),\\
&\{\check{\phi}(\vec{x},x^o),\Gamma(\vec{y},x^o)\} = \{\check{\pi}_\phi(\vec{x},x^o),\Gamma(\vec{y},x^o)\} = 0,
\end{aligned} \tag{11}$$

and again it amounts to add the Coulomb cloud to them.

Therefore, the physical Hamiltonian density after the symplectic decoupling of the gauge variables is

$$\begin{aligned}
\mathcal{H}^{(em)}_{phys}(x) = &\frac{1}{2}[\vec{\pi}^2_\perp(x) + \vec{B}^2(x)] + \check{\psi}^\dagger(x)\vec{\alpha}\cdot(i\vec{\partial} + e\vec{A}_\perp(x))\check{\psi}(x) + m\check{\bar{\psi}}(x)\check{\psi}(x) +\\
&+ \check{\pi}_\phi(x)\check{\pi}_{\phi*}(x) + [(\vec{\partial} + ie\vec{A}_\perp(x))\check{\phi}^*(x)]\cdot(\vec{\partial} - ie\vec{A}_\perp(x))\check{\phi}(x) +\\
&+ \lambda(\check{\phi}^*(x)\check{\phi}(x) - \phi_o^2)^2 +\\
&+ \frac{e^2}{2}[\check{\psi}^\dagger(x)\check{\psi}(x) - i(\check{\pi}_\phi(x)\check{\phi}(x) - \check{\pi}_{\phi*}(x)\check{\phi}^*(x))]\,\frac{1}{\triangle}\\
&\quad [\check{\psi}^\dagger(x)\check{\psi}(x) - i(\check{\pi}_\phi(x)\check{\phi}(x) - \check{\pi}_{\phi*}(x)\check{\phi}^*(x))].
\end{aligned} \tag{12}$$

This Hamiltonian density is analytic in the electric charge e but there is the nonlocal Coulomb interaction of the charged fields $\check{\psi},\check{\psi}^\dagger,\check{\phi},\check{\phi}^*$. See Refs. [1,8] and Section V for the reformulation on spacelike hypersurfaces to take care of Lorentz covariance.

The Hamilton equations imply

$$\begin{aligned}
\vec{\pi}_\perp(x) &= -\partial^o\vec{A}_\perp(x),\\
\check{\pi}_\phi(x) &= \partial^o\check{\phi}^*(x) - ie^2\check{\phi}^*(x)\frac{1}{\triangle}[\check{\psi}^\dagger(x)\check{\psi}(x) + i(\check{\pi}_{\phi^*}(x)\check{\phi}^*(x) - \check{\pi}_\phi(x)\check{\phi}(x))] =\\
&= \partial^o\check{\phi}^*(x) -
\end{aligned}$$

$$
\begin{aligned}
&- ie^2\check{\phi}^*(x)\frac{1}{\triangle + 2e^2\check{\phi}^*(x)\check{\phi}(x)}[\check{\psi}^\dagger(x)\check{\psi}(x) + i(\check{\phi}^*(x)\partial\check{\phi}(x) - \partial^o\check{\phi}^*(x)\check{\phi}(x))] \\
\check{\pi}_{\phi*}(x) &= \partial^o\check{\phi}(x) + ie^2\check{\phi}(x)\frac{1}{\triangle}[\check{\psi}^\dagger(x)\check{\psi}(x) + i(\check{\pi}_{\phi^*}(x)\check{\phi}^*(x) - \check{\pi}_\phi(x)\check{\phi}(x))] = \\
&= \partial^o\check{\phi}(x) + \\
&+ ie^2\check{\phi}(x)\frac{1}{\triangle + 2e^2\check{\phi}^*(x)\check{\phi}(x)}[\check{\psi}^\dagger(x)\check{\psi}(x) + i(\check{\phi}^*(x)\partial^o\check{\phi}(x) - \partial^o\check{\phi}^*(x)\check{\phi}(x))] \qquad (13)
\end{aligned}
$$

because we get

$$
\begin{aligned}
&\check{\pi}_{\phi^*}(x)\check{\phi}^*(x) - \check{\pi}_\phi(x)\check{\phi}(x) = \\
&\frac{1}{1 + 2e^2\check{\phi}^*(x)\check{\phi}(x)\frac{1}{\triangle}}[\partial^o\check{\phi}(x)\check{\phi}^*(x) - \partial^o\check{\phi}^*(x)\check{\phi}(x) + 2ie^2\check{\phi}^*(x)\check{\phi}(x)\frac{1}{\triangle}\check{\psi}^\dagger(x)\check{\psi}(x)] \qquad (14)
\end{aligned}
$$

Use has been done of the operator identity $\frac{1}{A}\frac{1}{1+B\frac{1}{A}} = \frac{1}{A}[1 - B\frac{1}{A} + B\frac{1}{A}B\frac{1}{A} - ...] = \frac{1}{A+B}$ (valid for B a small perturbation of A) for $A = \triangle$ and $B = \phi^*(x)\phi(x)$.

The nonlocal Lagrangian density generating $\mathcal{H}^{(em)}_{phys}(x)$ and describing only the electromagnetic phase is (see also Ref. [1])

$$
\begin{aligned}
\mathcal{L}^{(em)}_{phys}(x) = &\frac{1}{2}[\dot{\vec{A}}^2_\perp(x) - (\vec{\partial} \times \vec{A}_\perp(x))^2] + \\
&+ \check{\psi}^\dagger(x)[i\partial_o - \vec{\alpha}\cdot(i\vec{\partial} + e\vec{A}_\perp(x)) - \beta m]\check{\psi}(x) + \\
&+ \partial^o\check{\phi}^*(x)\partial^o\check{\phi}(x) - [(\vec{\partial} + ie\vec{A}_\perp(x))\check{\phi}^*(x)]\cdot(\vec{\partial} - ie\vec{A}_\perp(x))\check{\phi}(x) - \\
&- \lambda(\check{\phi}^*(x)\check{\phi}(x) - \phi_o^2)^2 - \\
&- \frac{e^2}{2}[\check{\psi}^\dagger(x)\check{\psi}(x) + i(\check{\phi}^*(x)\partial^o\check{\phi}(x) - \partial^o\check{\phi}^*(x)\check{\phi}(x))]\,\frac{1}{\triangle + 2e^2\check{\phi}^*(x)\check{\phi}(x)}\cdot \\
&\ [\check{\psi}^\dagger(x)\check{\psi}(x) + i(\check{\phi}^*(x)\partial^o\check{\phi}(x) - \partial^o\check{\phi}^*(x)\check{\phi}(x))], \qquad (15)
\end{aligned}
$$

The Hamilton equations of this phase are Eqs.(13) and

$$
\begin{aligned}
&\partial^o\vec{\pi}_\perp(\vec{x},x^o) \circeq \triangle\vec{A}_\perp(\vec{x},x^o) + e\check{\psi}^\dagger(\vec{x},x^o)\vec{\alpha}\check{\psi}(\vec{x},x^o) + \\
&+ ie[\check{\phi}^*(\vec{x},x^o)(\vec{\partial} - ie\vec{A}_\perp(\vec{x},x^o))\check{\phi}(\vec{x},x^o) - \check{\phi}(\vec{x},x^o)(\vec{\partial} + ie\vec{A}_\perp(\vec{x},x^o))\check{\phi}^*(\vec{x},x^o)
\end{aligned}
$$

$$
\begin{aligned}
&\partial^o\check{\pi}_\phi(\vec{x},x^o) \circeq (\vec{\partial} + ie\vec{A}_\perp(\vec{x},x^o))^2\check{\phi}^*(\vec{x},x^o) - 2\lambda\check{\phi}^*(\vec{x},x^o)[\check{\phi}^*(\vec{x},x^o)\check{\phi}(\vec{x},x^o) - \phi_o^2] + \\
&+ ie^2\check{\pi}_\phi(\vec{x},x^o)\frac{1}{\triangle}[\check{\psi}^\dagger(\vec{x},x^o)\check{\psi}(\vec{x},x^o) - i(\check{\pi}_\phi(\vec{x},x^o)\check{\phi}(\vec{x},x^o) - \check{\phi}_{\phi^*}(\vec{x},x^o)\check{\phi}^*(\vec{x},x^o))]
\end{aligned}
$$

$$\partial^o \check{\pi}_{\phi^*}(\vec{x},x^o) \stackrel{\circ}{=} (\vec{\partial}+ie\vec{A}_\perp(\vec{x},x^o))^2\check{\phi}^*(\vec{x},x^o) - 2\lambda\check{\phi}(\vec{x},x^o)[\check{\phi}^*(\vec{x},x^o)\check{\phi}(\vec{x},x^o)-\phi_o^2] -$$
$$- ie^2\check{\pi}_{\phi^*}(\vec{x},x^o)\frac{1}{\triangle}[\check{\psi}^\dagger(\vec{x},x^o)\check{\psi}(\vec{x},x^o) - i(\check{\pi}_\phi(\vec{x},x^o)\check{\phi}(\vec{x},x^o) - \check{\phi}_{\phi^*}(\vec{x},x^o)\check{\phi}^*(\vec{x},x^o))]$$

$$\partial^o\check{\psi}(\vec{x},x^o) \stackrel{\circ}{=} \vec{\alpha}\cdot(\vec{\partial}-ie\vec{A}_\perp(\vec{x},x^o))\check{\psi}(\vec{x},x^o) - im\beta\check{\psi}(\vec{x},x^o) -$$
$$- e^2\check{\psi}(\vec{x},x^o)\frac{1}{\triangle}[\check{\psi}^\dagger(\vec{x},x^o)\check{\psi}(\vec{x},x^o) - i(\check{\pi}_\phi(\vec{x},x^o)\check{\phi}(\vec{x},x^o) - \check{\phi}_{\phi^*}(\vec{x},x^o)\check{\phi}^*(\vec{x},x^o))], \qquad (16)$$

which imply the following Euler-Lagrange equations

$$\begin{aligned}
\Box\vec{A}_\perp(x) &\stackrel{\circ}{=} -e\check{\psi}^\dagger(x)\vec{\alpha}\check{\psi}(x) - \\
&- ie[\check{\phi}^*(x)(\vec{\partial}-ie\vec{A}_\perp(x))\check{\phi}(x) - \check{\phi}(x)(\vec{\partial}+ie\vec{A}_\perp(x))\check{\phi}^*(x) \\
&\ddot{\check{\phi}} - (\vec{\partial}-ie\vec{A}_\perp(x))^2\check{\phi}(x) \stackrel{\circ}{=} -2\lambda\check{\phi}^*(x)[\check{\phi}^*(x)\check{\phi}(x)-\phi_o^2] - \\
&- ie\{\dot{\check{\phi}}(x) + ie^2\check{\phi}(x)\frac{1}{\triangle+2e^2\check{\phi}^*(x)\check{\phi}(x)}[\check{\psi}^\dagger(x)\check{\psi}(x) + i(\check{\phi}^*(x)\dot{\check{\phi}}(x) - \dot{\check{\phi}}^*(x)\check{\phi}(x))]\} - \\
&- ie\partial^o\{\check{\phi}(x)\frac{1}{\triangle+2e^2\check{\phi}^*(x)\check{\phi}(x)}[\check{\psi}^\dagger(x)\check{\psi}(x) + i(\check{\phi}^*(x)\dot{\check{\phi}}(x) - \dot{\check{\phi}}^*(x)\check{\phi}(x))]\} \\
&(\partial^o - \vec{\alpha}\cdot(\vec{\partial}-ie\vec{A}_\perp(x)) + im\beta)\check{\psi}(x) \stackrel{\circ}{=} \\
&\stackrel{\circ}{=} -e^2\check{\psi}(x)\frac{1}{\triangle+2e^2\check{\phi}^*(x)\check{\phi}(x)}[\check{\psi}^\dagger(x)\check{\psi}(x) + i(\check{\phi}^*(x)\dot{\check{\phi}}(x) - \dot{\check{\phi}}^*(x)\check{\phi}(x))].
\end{aligned} \qquad (17)$$

These equations can be recovered from the Lagrangian density of Eq.(15) by using the identity $f\frac{\partial}{\partial V}\frac{1}{\triangle+V}f = -f\frac{1}{(\triangle+V)^2}f = -\frac{1}{\triangle+V}f\,\frac{1}{\triangle+V}f$ (modulo a surface term), where $f = \check{\psi}^\dagger\check{\psi} = i(\check{\phi}^*\dot{\check{\phi}} - \dot{\check{\phi}}^*\check{\phi})$ and $V = 2e^2\check{\phi}^*\check{\phi}$. The Higgs field $\phi(x)$ must be such that the operator $\triangle + 2e^2\check{\phi}^*(x)\check{\phi}(x)$ has no zero modes.

Eqs.(13) and (15) also give the reduction to Dirac's observables of a charged complex Klein-Gordon field interacting with the electromagnetic field.

III. THE HIGGS PHASE.

There are two methods to get this phase starting from the following parametrization of the Higgs fields [the value $\phi = 0$ is not covered by these radial coordinates; for the sake of simplicity we take a positive value $\phi_o > 0$ for the arbitrary symmetry breaking reference point in the set of minima of the potential: this set is spanned by varying an angular variable θ, so that θ is the would-be Goldstone boson; the symmetry group U(1) is broken and there is no residual stability group of the points of minimum]

$$\phi(x) = [\phi_o + \frac{1}{\sqrt{2}}H(x)]e^{ie\theta(x)} = \frac{1}{\sqrt{2}}(v + H(x))e^{ie\theta(x)}, \qquad v = \sqrt{2}\phi_o,$$
$$D^{(A)}_\mu \phi(x) = e^{ie\theta(x)}[\frac{1}{\sqrt{2}}\partial_\mu H(x) - ie(\phi_o + \frac{1}{\sqrt{2}}H(x))(A_\mu(x) - \partial_\mu \theta(x))], \tag{18}$$

so that the Lagrangian density becomes

$$\mathcal{L}(x) = -\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) + e^2(\phi_o + \frac{1}{\sqrt{2}}H(x))^2(A_\mu(x) - \partial_\mu\theta(x))(A^\mu(x) - \partial^\mu\theta(x)) +$$
$$+ \frac{1}{2}\partial_\mu H(x)\partial^\mu H(x) - \frac{\lambda}{2}H^2(x)(\frac{1}{\sqrt{2}}H(x) + 2\phi_o)^2 +$$
$$+ \frac{1}{2}\bar{\psi}(x)[\gamma^\mu(i\partial_\mu + eA_\mu(x)) - (i\overleftarrow{\partial_\mu} - eA_\mu(x))\,\gamma^\mu]\psi(x) - m\bar{\psi}(x)\psi(x). \tag{19}$$

The parametrization of Eq.(18) requires a restriction to Higgs fields which have no zeroes, namely $\phi^*(x)\phi(x) \neq 0$ $[H(x) \neq -v = -\sqrt{2}\phi_o]$, and with a nonsingular phase $\theta(x)$ because we assumed a trivial U(1) principal bundle. The analogue of the quantum statement of symmetry breaking, i.e. that the theory is invariant under a group G but not the ground state, is replaced by the choice of the parametrization (18) with a given ϕ_o, i.e. by the choice of a family of solutions of the Euler-Lagrange equations associated with Eq.(1) not invariant under U(1).

i) The canonical momenta coming from Eq.(19) are

$$\pi^o(x) = 0$$
$$\vec{\pi}(x) = \vec{E}(x)$$
$$\pi_H(x) = \partial^o H(x)$$
$$\pi_\theta(x) = 2e^2(\phi_o + \frac{1}{\sqrt{2}}H(x))^2(\partial_o\theta(x) - A_o(x))$$

$$\{H(\vec{x}, x^o), \pi_H(\vec{y}, x^o)\} = \{\theta(\vec{x}, x^o), \pi_\theta(\vec{y}, x^o)\} = \delta^3(\vec{x} - \vec{y}). \tag{20}$$

The resulting Dirac Hamiltonian density is

$$
\begin{aligned}
\mathcal{H}_D(x) &= \frac{1}{2}[\vec{\pi}^2(x) + \vec{B}^2(x)] + e^2(\phi_o + \frac{1}{\sqrt{2}}H(x))^2(\vec{A}(x) - \vec{\partial}\theta(x))^2 + \\
&+ \frac{1}{2}[\pi_H^2(x) + (\vec{\partial}H(x))^2] + \frac{\lambda}{2}H^2(x)(\frac{1}{\sqrt{2}}H(x) + 2\phi_o)^2 + \frac{\pi_\theta^2(x)}{4e^2(\phi_o + \frac{1}{\sqrt{2}}H(x))^2} + \\
&+ \psi^\dagger(x)\vec{\alpha}\cdot(i\vec{\partial} + e\vec{A}(x))\psi(x) + m\bar{\psi}(x)\psi(x) - \\
&- A_o(x)[-\vec{\partial}\cdot\vec{\pi}(x) + e\psi^\dagger(x)\psi(x) - \pi_\theta(x)] + \lambda_o(x)\pi^o(x) \qquad (21)
\end{aligned}
$$

and there are two first class constraints: $\pi^o(x) \approx 0$ and the Gauss law

$$
\hat{\Gamma}(x) = -\vec{\partial}\cdot\vec{\pi}(x) + e\psi^\dagger(x)\psi(x) - \pi_\theta(x) \approx 0, \qquad (22)
$$

which is now to be solved in $\pi_\theta(x)$. The pairs of conjugate gauge variables are now $A_o(x), \pi^o(x), \theta(x), \hat{\Gamma}(x)$, while the Dirac observables, having zero Poisson bracket with $\hat{\Gamma}(x)$, are

$$
\begin{aligned}
&\vec{A}'(x) = \vec{A}(x) - \vec{\partial}\theta(x), \\
&\vec{\pi}(x) = \vec{E}(x), \\
&\hat{\psi}(x) = e^{-ie\theta(x)}\psi(x), \\
&\hat{\psi}^\dagger(x) = e^{ie\theta(x)}\psi^\dagger(x), \\
&H(x), \\
&\pi_H(x). \qquad (23)
\end{aligned}
$$

and the Coulomb cloud of the electromagnetic phase has been now replaced by a Higgs (would-be Goldstone boson) cloud, which dresses the fermion fields and the vector field. In this way the would-be Goldstone boson (and the associated infrared singularities at the quantum level [9]) are "eaten" by the gauge boson which become massive. This is connected to the Gauss law [9], which is not trivial in presence of spontaneous symmetry breaking with the Higgs mechanism, as we shall see in the last Section.

After the symplectic decoupling without adding gauge-fixings, we get the following Hamiltonian density

$$
\begin{aligned}
\mathcal{H}^{(Higgs)}_{phys}(x) &= \frac{1}{2}[\vec{\pi}^2(x) + \vec{B}^2(x)] + \frac{1}{2}m^2_{em}(1 + \frac{|e|}{m_{em}}H(x))^2 \vec{A}'^2(x) + \\
&+ \frac{1}{2}[\pi^2_H(x) + (\vec{\partial}H(x))^2] + \frac{1}{2}m^2_H\, H^2(x)(1 + \frac{|e|}{2m_{em}}H(x))^2 + \\
&+ \hat{\psi}^\dagger(x)\vec{\alpha}\cdot(i\vec{\partial} + e\vec{A}'(x))\hat{\psi}(x) + m\hat{\bar{\psi}}(x)\hat{\psi}(x) + \frac{(\vec{\partial}\cdot\vec{\pi}(x) - e\hat{\psi}^\dagger(x)\hat{\psi}(x))^2}{2m^2_{em}(1 + \frac{|e|}{m_{em}}H(x))^2}, \qquad (24)
\end{aligned}
$$

which is local but not analytic in the electric charge e or, by replacing ϕ_o with the electromagnetic mass $m_{em} = \sqrt{2}|e|\phi_o = |e|v$, in the sum of the mass produced by the spontaneous symmetry breaking and the residual Higgs field, whose mass is $m_H = 2\phi_o\sqrt{\lambda}$ [so that $\phi_o = m_{em}/\sqrt{2}|e|$ and $\lambda = e^2m^2_H/2m^2_{em}$]. From Eq.(16) we get $\pi_\theta = (m_{em} + |e|H)^2(\partial_o\theta + A_o)$. Let us remark that in those points x^μ where $H(x) = -m_{em}/|e| = -\sqrt{2}\phi_o$ [which were excluded to exist not to have problems with the origin of the radial coordinates of Eq.(18)] we would recover massless electromagnetism, so that the numerator of the self-energy term in Eq.(24) must vanish, being the Gauss law of the massless theory. Therefore we should not have a singularity in these points, but new physical effects as shown in Section IV.

Let us remark that the self-energy appearing in Eq.(24) is local and that, in presence of fermion fields, it contains a 4 fermion interaction, which has appeared from the nonperturbative solution of the Gauss law and which is a further obstruction to the renormalizability of the reduced theory (equivalent to the unitary gauge, but without having added any gauge-fixing), which already fails in the unitary physical gauge due to the massive vector boson propagator not fulfilling the power counting rule; as said in Ref. [10], this is due to the fact that the field-dependent gauge transformation relating $\vec{A}$ and $\vec{A}'$ in Eq.(23) is not unitarily implementable. It is interesting to note that all the interaction terms of the residual Higgs field $H(x)$ in Eq.(24) show that it couples to the ratio $|e|/m_{em}$.

Again the lack of manifest Lorentz covariance can be taken care of by reformulating the theory on spacelike hypersurfaces, as shown in Section IV.

Since one has

$$
\partial^o A'^i(\vec{x}, x^o) = \{A'^i(\vec{x}, x^o), \int d^3y \mathcal{H}^{(Higgs)}_{phys}(\vec{y}, x^o)\} =
$$

$$= -\pi^i(\vec{x},x^o) + \partial^i \frac{\vec{\partial}\cdot\vec{\pi}(\vec{x},x^o) - e\hat{\psi}^\dagger(\vec{x},x^o)\hat{\psi}(\vec{x},x^o)}{(m_{em}+|e|H(x))^2}$$

$$\begin{aligned} \Rightarrow\ \pi^i\,(x) &= -\partial^o A^{'i}(x) - \\ &- \partial^i \frac{1}{\triangle + (m_{em}+|e|H(x))^2}\,[\vec{\partial}\cdot\partial^o\vec{A}^{''}(x) + e\hat{\psi}^\dagger(x)\hat{\psi}(x)], \end{aligned} \tag{25}$$

we get a nonlocal Lagrangian density describing only the Higgs phase

$$\begin{aligned} \mathcal{L}^{(Higgs)}_{phys}(x) &= \hat{\psi}^\dagger(x)[i\partial^o - \vec{\alpha}\cdot(i\vec{\partial}+e\vec{A}^{''}(x)) - \beta m]\hat{\psi}(x) + \\ &+ \frac{1}{2}[(\partial^o\vec{A}^{''}(x))^2 - [\vec{\partial}\cdot\partial^o\vec{A}^{''}(x) + e\hat{\psi}^\dagger(x)\hat{\psi}(x)] \\ &\frac{1}{\triangle + m^2_{em}(1+\frac{|e|}{m_{em}}H(x))^2}[\vec{\partial}\cdot\partial^o\vec{A}^{''}(x) + e\hat{\psi}^\dagger(x)\hat{\psi}(x)]] - \frac{1}{2}\vec{B}^2(x) - \\ &- \frac{1}{2}m^2_{em}(1+\frac{|e|}{m_{em}}H(x))^2\vec{A}^{''2}(x) + \\ &+ \frac{1}{2}\partial_\mu H(x)\partial^\mu H(x) - \frac{1}{2}m^2_H H^2(x)(1+\frac{|e|}{2m_{em}}H(x))^2. \end{aligned} \tag{26}$$

We see that the potential problems of the Hamiltonian formulation at the points where $H(x) = -m_{em}/|e| = -\sqrt{2}\phi_o$ are now replaced bythe requirement that the operator $\triangle + m^2_{em}(1+\frac{|e|}{m_{em}}H(x))^2$ must not have zero modes.

ii) Since the Dirac observables $\vec{A}^{''} = \vec{A} - \vec{\partial}\theta$ are obtained from $\vec{A}$ with a θ-field-dependent gauge transformation, which is the space part of the gauge transformation $A_\mu \mapsto A^{'}_\mu = A_\mu - \partial_\mu\theta$, $\phi(x) \mapsto \phi_o$, $\psi(x) \mapsto \hat{\psi}(x) = e^{-ie\theta(x)}\psi(x)$, used to go to the "unitary gauge" [11,12], we can do this gauge transformation in the gauge invariant Lagrangian density of Eq.(19) to get

$$\begin{aligned} \mathcal{L}(x) &= \mathcal{L}^{'}(x) = \\ &= -\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) + \frac{1}{2}m^2_{em}(1+\frac{|e|}{m_{em}}H(x))^2 A^{'2}(x) + \\ &+ \frac{1}{2}\partial_\mu H(x)\partial^\mu H(x) - \frac{1}{2}m^2_H H^2(x)(1+\frac{|e|}{2m_{em}}H(x))^2 + \\ &+ \hat{\psi}^\dagger(x)\gamma^\mu(i\partial_\mu + eA^{'}_\mu(x))\hat{\psi}(x) - m\hat{\bar{\psi}}(x)\hat{\psi}(x). \end{aligned} \tag{27}$$

Now $\mathcal{L}^{'}(x)$ depends on $A^{'}_\mu, H, \hat{\psi}, \hat{\psi}^\dagger$, but not on θ. The new momenta are

$$\pi^o(x) = 0,$$

$$\vec{\pi}(x) = \vec{E}(x),$$

$$\pi_H(x) = \partial^o H(x), \tag{28}$$

and the new Dirac Hamiltonian density is

$$\begin{aligned}\mathcal{H}_D^{'}(x) &= \frac{1}{2}[\vec{\pi}^2(x) + \vec{B}^2(x)] - \frac{1}{2}m_{em}^2(1 + \frac{|e|}{m_{em}}H(x))^2[A_o^{'2}(x) - \vec{A}^{'2}(x)] + \\ &+ \hat{\psi}^\dagger(x)\vec{\alpha}\cdot(i\vec{\partial} + e\vec{A}^{'}(x))\hat{\psi}(x) + m\hat{\bar{\psi}}(x)\hat{\psi}(x) - \\ &- A_o^{'}(x)[-\vec{\partial}\cdot\vec{\pi}(x) + e\hat{\psi}^\dagger(x)\hat{\psi}(x)] + \frac{1}{2}[\pi_H^2(x) + (\vec{\partial}H(x))^2] + \\ &+ \frac{1}{2}m_H^2 H^2(x)(1 + \frac{|e|}{2m_{em}}H(x))^2 + \lambda_o(x)\pi^o(x). \end{aligned} \tag{29}$$

Now the time constancy of the primary constraint $\pi^o(x) \approx 0$ generates the $A_o^{'}$-dependent secondary constraint

$$\zeta(x) = (m_{em} + |e|H(x))^2 A_o^{'}(x) - \vec{\partial}\cdot\vec{\pi}(x) + e\hat{\psi}^\dagger(x)\hat{\psi}(x) \approx 0. \tag{30}$$

The time constancy of $\zeta(x) \approx 0$ determines the Dirac multiplier $\lambda_o(x)$, so that now $\pi^o(x) \approx 0, \zeta(x) \approx 0$ are a pair of second class constraints eliminating $A_o^{'}(x)$ and $\pi^o(x)$ by going to Dirac brackets. The substitution of the value of $A_o^{'}(x)$ given by $\zeta(x) \equiv 0$ into Eq.(29) reproduces Eq.(24).

Let us remark that this mechanism of second class constraints is the same which acts in the search of Dirac's observables of the standard massive vector field described by the Lagrangian density

$$\mathcal{L}(x) = -\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) + \frac{1}{2}M^2 A_\mu(x)A^\mu(x), \tag{31}$$

which is not gauge invariant under U(1) local gauge transformations. Its Euler-Lagrange equations $\partial_\nu F^{\nu\mu}(x) + M^2 A^\mu(x) \overset{\circ}{=} 0$ imply $(\Box + M^2)A_\mu(x) \overset{\circ}{=} 0$ and $\partial^\mu A_\mu(x) \overset{\circ}{=} 0$. The canonical momenta are $\pi^o(x) = 0$ and $\vec{\pi}(x) = \vec{E}(x)$. Associated with the primary constraint $\pi^o(x) \approx 0$ there is the following gauge transformation $\delta A_o(x) = \Lambda(x)$ [$\Lambda(x)$ arbitrary function], $\delta\vec{A}(x) = 0$, under which the Lagrangian density is quasi-invariant,

$\delta\mathcal{L}(x) = (\partial_k F^{ko}(x) + M^2 A^o(x))\delta A_o(x) \stackrel{\circ}{=} 0$, as it must be with second-class primary constraints [13,1]. The canonical Dirac Hamiltonian density is $\mathcal{H}_D(x) = \frac{1}{2}[\vec{\pi}^2(x) + \vec{B}^2(x)] - \frac{M^2}{2}[A_o^2(x) - \vec{A}^2(x)] + A_o(x)\vec{\partial}\cdot\vec{\pi}(x) + \lambda_o(x)\pi^o(x)$. The time constancy of $\pi^0(x) \approx 0$ generates the secondary constraint $\zeta'(x) = M^2 A_o(x) - \vec{\partial}\cdot\vec{\pi}(x) \approx 0$ and its time constancy determines the Dirac multiplier $\lambda_o(x) \stackrel{\circ}{=} \vec{\partial}\cdot\vec{A}(x)$ [i.e. $\partial^\mu A_\mu(x) \stackrel{\circ}{=} 0$, because $\lambda_o(x) \stackrel{\circ}{=} \partial^o A_o(x)$]. The Dirac observables of the model are $\vec{A}(x), \vec{\pi}(x)$, and the final physical Hamiltonian and Lagrangian densities are

$$\mathcal{H}^{(mass)}_{phys}(x) = \frac{1}{2}[\vec{\pi}^2(x) + \vec{B}^2(x)] + \frac{1}{2}M^2\vec{A}^2(x) + \frac{(\vec{\partial}\cdot\vec{\pi}(x))^2}{2M^2}$$

$$\begin{aligned}\mathcal{L}^{(mass)}_{phys}(x) &= \frac{1}{2}[(\partial^o\vec{A}(x))^2 - \vec{\partial}\cdot\partial^o\vec{A}(x)\frac{1}{\triangle + M^2}\vec{\partial}\cdot\partial^o\vec{A}(x)] - \\ &- \frac{1}{2}\vec{B}^2(x) - \frac{1}{2}M^2\vec{A}^2(x).\end{aligned} \tag{32}$$

From the Hamilton equations $\dot{A}^i(x) \stackrel{\circ}{=} -(\delta^{ij} - \frac{\partial^i\partial^j}{M^2})\pi^j(x)$, $\dot{\pi}^i(x) \stackrel{\circ}{=} (\triangle + M^2)(\delta^{ij} + \frac{\partial^i\partial^j}{\triangle + M^2})A^j(x)$, we get $\pi^i = -(\delta^{ij} + \frac{\partial^i\partial^j}{\triangle+M^2})\partial^o A^j$ and the Euler-Lagrange equations

$$\begin{aligned}&(\Box + M^2)(\delta^{ij} + \frac{\partial^i\partial^j}{\triangle + M^2})A^j(x) \stackrel{\circ}{=} 0\\ \Rightarrow\ &\frac{1}{\triangle + M^2}(\Box + M^2)\vec{\partial}\cdot\vec{A}(x) \stackrel{\circ}{=} 0, \qquad (\Box + M^2)\vec{A}(x) \stackrel{\circ}{=} 0.\end{aligned} \tag{33}$$

As noted in Ref. [14], one can consistently eliminate the residual Higgs field $H(x)$ at the Hamiltonian level, even if in the Abelian theory it is physically relevant being connected to amplitude effects coming from the condensate (Cooper pairs) simulated by the scalar fields. The elimination can be done by adding to the Hamiltonian density of the Higgs phase, Eq.(24), or to the Lagrangian density (26), a term $\mu(x)H(x)$ with $\mu(x)$ a Lagrange multiplier. This would imply a new holonomic constraint $H(x) \approx 0$ [in some sense it would correspond to $m_H \to \infty$; one could also require $H(x) - h \approx 0$ with $h = const.$] whose time constancy at the Hamiltonian level would generate the secondary constraint $\pi_H(x) \approx 0$. The time constancy of this secondary constraint would determine the multiplier $\mu(x)$, so that the two constraints turn out to be second class. By going to Dirac brackets, we would obtain a theory without residual Higgs fields described by the Hamiltonian density of Eq.(24) evaluated at $H(x) = \pi_H(x) \equiv 0$

$$\tilde{\mathcal{H}}^{(Higgs)}_{phys}(x) = \frac{1}{2}[\vec{\pi}^2(x) + \vec{B}^2(x)] + \frac{1}{2} m^2_{em} \vec{A}^{\prime 2}(x) + \hat{\psi}^\dagger(x)\vec{\alpha}\cdot(i\vec{\partial} + e\vec{A}^{\prime}(x))\hat{\psi}(x) + + m\hat{\bar{\psi}}(x)\hat{\psi}(x) + \frac{(\vec{\partial}\cdot\vec{\pi}(x) - e\hat{\psi}^\dagger(x)\hat{\psi}(x))^2}{2m^2_{em}}. \tag{34}$$

to be compared with Eq.(32). The elimination of $H(x)$ reproduces the massive vector theory and can also be thought as a limiting classical result of the so-called "triviality problem" [triviality of the $\lambda\phi^4$ theory [15]], which however would imply a quantization (but how?) of the Higgs phase alone without the residual Higgs field, so that also its quantum fluctuations would be absent (instead they are the main left quantum effect in the limit $m_H \rightarrow \infty$, which is known to produce [16], in the non-Abelian case, a gauge theory coupled to a nonlinear σ-model, equivalent [17] to a massive Yang-Mills theory).

The physical Hamiltonian of Eq.(26) implies the Hamilton equations

$$\begin{aligned}
\partial^o \vec{A}^{\prime}(\vec{x},x^o) &\overset{\circ}{=} -\vec{\pi}(\vec{x},x^o) + \vec{\partial}\, \frac{\vec{\partial}\cdot\vec{\pi}(\vec{x},x^o) - e\hat{\psi}^\dagger(\vec{x},x^o)\hat{\psi}(\vec{x},x^o)}{(m_{em} + |e|H(\vec{x},x^o))^2}\\
\partial^o \vec{\pi}(\vec{x},x^o) &\overset{\circ}{=} (m_{em} + |e|H(\vec{x},x^o))^2 \vec{A}^{\prime}(\vec{x},x^o) + e\hat{\psi}^\dagger(\vec{x},x^o)\vec{\alpha}\hat{\psi}(\vec{x},x^o) + \\
&+ \triangle\vec{A}^{\prime}(\vec{x},x^o) + \vec{\partial}(\vec{\partial}\cdot\vec{A}^{\prime}(\vec{x},x^o))\\
\partial^o H(\vec{x},x^o) &\overset{\circ}{=} \pi_H(\vec{x},x^o)\\
\partial^o \pi_H(\vec{x},x^o) &\overset{\circ}{=} -|e|(m_{em} + |e|H(\vec{x},x^o))\vec{A}^{\prime 2}(\vec{x},x^o) - \triangle H(\vec{x},x^o) - \\
&- m^2_H H(\vec{x},x^o)(1 + \frac{|e|}{2m_{em}} H(\vec{x},x^o))^2 - \frac{|e| m^2_H}{2m_{em}} H^2(\vec{x},x^o)(1 + \frac{|e|}{2m_{em}} H(\vec{x},x^o)) + \\
&+ |e| \frac{(\vec{\partial}\cdot\vec{\pi}(\vec{x},x^o) - e\hat{\psi}^\dagger(\vec{x},x^o)\hat{\psi}(\vec{x},x^o))^2}{(m_{em} + |e|H(\vec{x},x^o))^3}\\
\partial^o \hat{\psi}(\vec{x},x^o) &\overset{\circ}{=} \vec{\alpha}\cdot(\vec{\partial} - ie\vec{A}^{\prime}(\vec{x},x^o))\hat{\psi}(\vec{x},x^o) - im\gamma^o\hat{\psi}(\vec{x},x^o) + \\
&+ ie\hat{\psi}(\vec{x},x^o)\, \frac{\vec{\partial}\cdot\vec{\pi}(\vec{x},x^o) - e\hat{\psi}^\dagger(\vec{x},x^o)\hat{\psi}(\vec{x},x^o)}{(m_{em} + |e|H(\vec{x},x^o))^2}.
\end{aligned} \tag{35}$$

From them we get $\frac{\vec{\partial}\cdot\vec{\pi} - e\hat{\psi}^\dagger\hat{\psi}}{(m_{em}+|e|H)^2} = -\frac{1}{\triangle + (m_{em}+|e|H)^2}(\partial^o\vec{\partial}\cdot\vec{A}^{\prime} + e\hat{\psi}^\dagger\hat{\psi})$, $\vec{\pi} = -\partial^o\vec{A}^{\prime} - \vec{\partial}\,\frac{1}{\triangle + (m_{em}+|e|H)^2}(\partial^o\vec{\partial}\cdot\vec{A}^{\prime} + e\hat{\psi}^\dagger\hat{\psi})$ and the following Euler-Lagrange equations

$$\{\Box + (m_{em} + |e|H)^2\}\vec{A}^{\prime}(x) + [\vec{\partial} + \partial^o \frac{1}{\triangle + (m_{em} + |e|H)^2}\partial^o]\vec{\partial}\cdot\vec{A}^{\prime}(x) +$$

$$+ (m_{em} + |e|H(x))^2 \vec{A}'(x) \stackrel{\circ}{=} - e\hat{\psi}^\dagger(x)\vec{\alpha}\hat{\psi}(x) - \partial^o \vec{\partial} \frac{1}{\triangle + (m_{em} + |e|H)^2} (e\hat{\psi}^\dagger(x)\hat{\psi}(x))$$

$$\Box H(x) \stackrel{\circ}{=} -|e|(m_{em} + |e|H(x))\vec{A}'^2(x) - m_H^2 H(x)(1 + \frac{|e|}{2m_{em}} H(x))^2 -$$

$$- \frac{|e|m_H^2}{2m_{em}} H^2(x)(1 + \frac{|e|}{2m_{em}} H(x)) +$$

$$+ |e|(m_{em} + |e|H(x))[\frac{1}{\triangle + (m_{em} + |e|H)^2} (\partial^o \vec{\partial} \cdot \vec{A}'(x) + e\hat{\psi}^\dagger(x)\hat{\psi}(x))]^2$$

$$(i\partial^o - \vec{\alpha} \cdot (i\vec{\partial} + e\vec{A}'(x)) - m\gamma^o)\hat{\psi}(x) \stackrel{\circ}{=}$$

$$\stackrel{\circ}{=} e\hat{\psi}(x) \frac{1}{\triangle + (m_{em} + |e|H)^2} (\partial^o \vec{\partial} \cdot \vec{A}'(x) + e\hat{\psi}^\dagger(x)\hat{\psi}(x)). \quad (36)$$

which can be recovered from the Lagrangian density of Eq.(23) by using the same identity given at the end of Section 2 with $f = \partial^o \vec{\partial} \cdot \vec{A}' + e\hat{\psi}^\dagger \hat{\psi}$ and $V = (m_{em} + |e|H)^2$.

We do not know how to solve the coupled Eqs.(36). Let us make two comments on the case without fermions, so that the third line of Eqs.(36) is missing. By turning off the coupling constant e the first of Eqs.(36) becomes consistent with the massive vector theory, while the second one becomes a Klein-Gordon equation for $H(x)$. By putting $H(x) = 0$ in Eqs.(36) (limit $m_H \rightarrow \infty$), the first line still reproduces the massive vector theory, but the second line becomes the restriction $|e|m_{em}[(\frac{\partial^o}{\triangle + m_{em}^2} \vec{\partial} \cdot \vec{A}')^2 - \vec{A}'^2] \stackrel{\circ}{=} 0$ on the space of solutions of Eq.(33) [in this approximation, with $\vec{A}' = \vec{A}'_\perp - \frac{\vec{\partial}}{\triangle} \vec{\partial} \cdot \vec{A}'$, the first of Eqs.(36) becomes $(\Box + m_{em}^2)\vec{A}'_\perp \stackrel{\circ}{=} 0$, $(\partial_o^2 \frac{m_{em}^2}{\triangle + m_{em}^2} + m_{em}^2)\vec{\partial} \cdot \vec{A}' \stackrel{\circ}{=} 0$]. Therefore a weak nearly constant Higgs field (strongly interacting symmetry breaking sector for $m_h \rightarrow \infty$) influences the longitudinal polarization of the massive vector field; this is the main difference from the massive vector theory introduced by the Higgs mechanism.

Let us remark that, while in the electromagnetic phase the self-energy term in the physical Hamiltonian (12) is nonlocal, the self-energy term in Eq.(24) of the Higgs phase is local implying a local four-fermion coupling. This feature is common to the physical Hamiltonian (32) of the standard massive vector field and to Eq.(34). There is a not-manifestly Lorentz covariant modification of Eq.(31) involving only the nonphysical variable $A_o(x)$, namely

$$\mathcal{L}'(x) = -\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x) + \frac{1}{2}[M^2 A_o^2(x) + \vec{\partial} A_o(x) \cdot \vec{\partial} A_o(x)] - \frac{1}{2} M^2 \vec{A}^2(x), \quad (37)$$

which solves this problem and which can be made Lorentz covariant by its reformulation on spacelike hypersurfaces as shown in Section V. The new Euler-Lagrange equations and the Dirac Hamiltonian density are respectively [the canonical momenta are the same of the massive vector theory]

$$\partial_\nu F^{\nu\mu}(x) + M^2 A^\mu(x) + \eta^{\mu o}\triangle A^o(x) \overset{\circ}{=} 0$$
$$\Rightarrow \quad (\Box + M^2)A^\mu(x) - \partial^\mu\partial_\nu A^\nu(x) + \eta^{\mu o}\triangle A^o(x) \overset{\circ}{=} 0$$

$$\mathcal{H}^{'}(x) = \frac{1}{2}[\vec{\pi}^2(x) + \vec{B}^2(x)] - \frac{1}{2}A_o(x)(\triangle + M^2)A_o(x) + \frac{1}{2}M^2\vec{A}^2(x) +$$
$$+A_o(x)\vec{\partial}\cdot\vec{\pi}(x) + \lambda_o(x)\pi^o(x). \tag{38}$$

The time constancy of the primary constraint $\pi^o(x) \approx 0$ produces the secondary one $\tilde{\zeta}(x) = (\triangle + M^2)A_o(x) - \vec{\partial}\cdot\vec{\pi}(x) \approx 0$, whose time constancy determines $\lambda_o(x) \approx \frac{M^2}{\triangle + M^2}\vec{\partial}\cdot\vec{A}(x)$ consistently with the Euler-Lagrange equations and with $\lambda_o(x) \overset{\circ}{=} \dot{A}_o(x)$. By eliminating $A_o(x), \pi^o(x)$ with the pair of second class constraints, we arrive at the physical Hamiltonian density

$$\mathcal{H}^{'}_{phys}(x) = \frac{1}{2}[\vec{\pi}^2(x) + \vec{B}^2(x)] + \frac{1}{2}M^2\vec{A}^2(x) + \frac{1}{2}\vec{\partial}\cdot\vec{\pi}(x)\,\frac{1}{\triangle + M^2}\,\vec{\partial}\cdot\vec{\pi}(x). \tag{39}$$

A gauge transformation $\delta A_o(x)$, generated by $\pi^o(x) \approx 0$, would produce a weak quasi-invariance [18] $\delta\mathcal{L}(x) = -[\partial_\nu F^{\nu o}(x) + (\triangle + M^2)A^o(x)]\delta A_o(x) = -[(2\triangle + M^2)A^o(x) + \partial^o\vec{\partial}\cdot\vec{A}(x)]\delta A_o(x) \overset{\circ}{=} 0$, i.e. $\delta\mathcal{L}(x)$ vanishes by using the acceleration independent Euler-Lagrange equation corresponding to the Gauss law.

The Hamilton equations $\dot{A}^i(x) \overset{\circ}{=} -\pi^i(x) + \frac{\partial^i}{\triangle + M^2}\,\vec{\partial}\cdot\vec{\pi}(x)$, $\dot{\pi}^i(x) \overset{\circ}{=} (\triangle + M^2)A^i(x) + \partial^i\vec{\partial}\cdot\vec{A}(x)$, imply $\pi^i(x) = -(\delta^{ij} + \frac{\partial^i\partial^j}{2\triangle + M^2})\dot{A}^j(x)$ and

$$(\Box + M^2)A^i(x) + \partial^i[\frac{\partial_o^2}{2\triangle + M^2} + 1]\vec{\partial}\cdot\vec{A}(x) \overset{\circ}{=} 0$$
$$\Rightarrow \quad \frac{1}{2\triangle + M^2}[(\triangle + M^2)(\Box + M^2) - \triangle^2]\vec{\partial}\cdot\vec{A}(x) \overset{\circ}{=} 0, \qquad (\Box + M^2)A^i_\perp(x) \overset{\circ}{=} 0, \tag{40}$$

where $A^i = A^i_\perp - \frac{\partial^i}{\triangle}\vec{\partial}\cdot\vec{A}$.

In this way one gets a nonlocal self-energy (avoiding a local four-fermion interaction when fermions are present) with the correct massive Green function $e^{-M|\vec{x}-\vec{y}|}/4\pi|\vec{x}-\vec{y}|$. The transverse field still obeys the wave equation, while the longitudinal field has a modified wave equation [it can also be written in the form $[\partial_o^2 \frac{\triangle+M^2}{2\triangle+M^2} + M^2]\vec{\partial}\cdot\vec{A}(x) \circeq 0]$.

Therefore, both the Higgs field, Eq.(36) without fermions, and this modification of the standard massive theory produce complicated equations of motion for the longitudinal part of the vector field.

Finally, to introduce a similar effect in the Lagrangian density (1), we should add a term $\frac{1}{2}\vec{\partial}A'_o(x)\cdot\vec{\partial}A'_o(x)$ to the Lagrangian density (27) in the unitary gauge, because in this way the secondary constraint (30) would become $\zeta(x) = [\triangle + (m_{em} + |e|H(x))^2]A'_o(x) - \vec{\partial}\cdot\vec{\pi}(x) + e\hat{\psi}^\dagger(x)\hat{\psi}(x) \approx 0$ and the last term in the physical Hamiltonian (24) would be replaced by $\frac{1}{2}[\vec{\partial}\cdot\vec{\pi}(x) - e\hat{\psi}^\dagger(x)\hat{\psi}(x)]\frac{1}{\triangle+m_{em}^2(1+\frac{|e|}{m_{em}}H(x))^2}[\vec{\partial}\cdot\vec{\pi}(x) - e\hat{\psi}^\dagger(x)\hat{\psi}(x)]$ avoiding the local 4-fermion interaction. Therefore, the Lagrangian density (1) should be replaced by $\mathcal{L}(x) + \frac{1}{2}\vec{\partial}A_o(x)\cdot\vec{\partial}A_o(x)$, and we get the following modification of Eqs.(35), (36)

$$
\begin{aligned}
\partial^o \vec{A}''(\vec{x},x^o) &\circeq -\vec{\pi}(\vec{x},x^o) + \vec{\partial}\,\frac{1}{\triangle + (m_{em} + |e|H(\vec{x},x^o))^2}\,[\vec{\partial}\cdot\vec{\pi}(\vec{x},x^o) - e\hat{\psi}^\dagger(\vec{x},x^o)\hat{\psi}(\vec{x},x^o)\,]\\
\partial^o \vec{\pi}(\vec{x},x^o) &\circeq (m_{em} + |e|H(\vec{x},x^o))^2\vec{A}''(\vec{x},x^o) + e\hat{\psi}^\dagger(\vec{x},x^o)\vec{\alpha}\hat{\psi}(\vec{x},x^o) +\\
&\quad + \triangle\vec{A}''(\vec{x},x^o) + \vec{\partial}(\vec{\partial}\cdot\vec{A}''(\vec{x},x^o))\\
\partial^o H(\vec{x},x^o) &\circeq \pi_H(\vec{x},x^o)\\
\partial^o \pi_H(\vec{x},x^o) &\circeq -|e|(m_{em} + |e|H(\vec{x},x^o))\vec{A}''^2(\vec{x},x^o) - \triangle H(\vec{x},x^o) -\\
&\quad - m_H^2 H(\vec{x},x^o)(1 + \frac{|e|}{2m_{em}}H(\vec{x},x^o))^2 - \frac{|e|m_H^2}{2m_{em}}H^2(\vec{x},x^o)(1 + \frac{|e|}{2m_{em}}H(\vec{x},x^o)) +\\
&\quad + |e|(m_{em} + |e|H(\vec{x},x^o))\,[\vec{\partial}\cdot\vec{\pi}(\vec{x},x^o) - e\hat{\psi}^\dagger(\vec{x},x^o)\hat{\psi}(\vec{x},x^o)]\\
&\qquad \frac{1}{(\triangle + (m_{em} + |e|H(\vec{x},x^o))^2)^2}[\vec{\partial}\cdot\vec{\pi}(\vec{x},x^o) - e\hat{\psi}^\dagger(\vec{x},x^o)\hat{\psi}(\vec{x},x^o)]\\
\partial^o \hat{\psi}(\vec{x},x^o) &\circeq \vec{\alpha}\cdot(\vec{\partial} - ie\vec{A}''(\vec{x},x^o))\hat{\psi}(\vec{x},x^o) - im\gamma^o\hat{\psi}(\vec{x},x^o) +\\
&\quad + ie(m_{em} + |e|H(\vec{x},x^o))\hat{\psi}(\vec{x},x^o)\\
&\qquad \frac{1}{\triangle + (m_{em} + |e|H(\vec{x},x^o))^2}\,[\vec{\partial}\cdot\vec{\pi}(\vec{x},x^o) - e\hat{\psi}^\dagger(\vec{x},x^o)\hat{\psi}(\vec{x},x^o)]
\end{aligned}
$$

$$
\begin{aligned}
&\{\Box + (m_{em} + |e|H)^2\}\vec{A}^{'}(x) + \vec{\partial}\,\vec{\partial}\cdot\vec{A}^{'}(x) \circeq \\
&\quad \circeq -e\hat{\psi}^{\dagger}(x)\vec{\alpha}\hat{\psi}(x) - \partial^o\vec{\partial}\frac{\partial^o\,\vec{\partial}\cdot\vec{A}^{'}(x) + e\hat{\psi}^{\dagger}(x)\hat{\psi}(x)}{(m_{em} + |e|H)^2} \\
&\Box H(x) \circeq -|e|(m_{em} + |e|H(x))\vec{A}^{'2}(x) - m_H^2 H(x)\left(1 + \frac{|e|}{2m_{em}}H(x)\right)^2 - \\
&\quad - \frac{|e|m_H^2}{2m_{em}}H^2(x)\left(1 + \frac{|e|}{2m_{em}}H(x)\right) + \\
&\quad + |e|(m_{em} + |e|H(x))\left[\frac{1}{\triangle + (m_{em} + |e|H)^2}\frac{\partial^o\vec{\partial}\cdot\vec{A}^{'}(x) + e\hat{\psi}^{\dagger}(x)\hat{\psi}(x))}{(m_{em} + |e|H(x))^2}\right] \\
&\qquad (\triangle + (m_{em} + |e|H(x))^2)^2\left[\frac{1}{\triangle + (m_{em} + |e|H)^2}\frac{\partial^o\vec{\partial}\cdot\vec{A}^{'}(x) + e\hat{\psi}^{\dagger}(x)\hat{\psi}(x))}{(m_{em} + |e|H(x))^2}\right] \\
&(i\partial^o - \vec{\alpha}\cdot(i\vec{\partial} + e\vec{A}^{'}(x)) - m\gamma^o)\hat{\psi}(x) \circeq \\
&\quad \circeq e\hat{\psi}(x)\frac{\partial^o\vec{\partial}\cdot\vec{A}^{'}(x) + e\hat{\psi}^{\dagger}(x)\hat{\psi}(x))}{m_{em} + |e|H(x)}.
\end{aligned}
\tag{41}
$$

In the weak nearly constant Higgs field approximation, the analogue of the first of Eqs.(36), with $\vec{A}^{'} = \vec{A}^{'}_{\perp} - \frac{\vec{\partial}}{\triangle}\vec{\partial}\cdot\vec{A}^{'}$, becomes $(\Box + m_{em}^2)\vec{A}^{'}_{\perp} \circeq 0$, $[\partial_o^2 \frac{\triangle + m_{em}^2}{2\triangle + m_{em}^2} + m_{em}^2]\vec{\partial}\cdot\vec{A}^{'} \circeq 0$, while the second of Eqs.(36) gives the restriction $|e|m_{em}[\frac{\partial^o}{\triangle + m_{em}^2}\vec{\partial}\cdot\vec{A}^{'}\,(1 + \frac{\triangle}{m_{em}^2})^2\,\frac{\partial^o}{\triangle + m_{em}^2}\vec{\partial}\cdot\vec{A}^{'} - \vec{A}^{'2}] \circeq 0$.

IV. NIELSEN-OLESEN VORTICES

In this paper we have considered only trivial U(1) principal bundles over Minkowski spacetime (or better over its fixed x^o slices R^3), avoiding monopole configurations [19]. As shown for instance in Ref. [20], in presence of monopoles one has a nontrivial U(1) principal bundle over $M^3 = R^3 - \{set\, of\, points\, where\;\, monopoles\, are\, located\}$ [so that $\pi_o(M^3) = \pi_1(M^3) = 0$ but $\pi_2(M^3) \neq 0$, with $\pi_k(M^3)$ being the k-th homotopy group of M^3]. Therefore the gauge potentials A_μ (cross sections of the U(1) principal bundle) and the Lagrangian density of Eq.(1) cannot be globally defined, since there are no global cross sections; instead there is a well defined Hamiltonian formalism. In our formalism potential problems in the Higgs phase arise in those points where $H(x) = -m_{em}/|e| = -\sqrt{2}\phi_o$, in which the theory is nonanalytic and could have, a priori, essential singularities. See Ref. [21] for a possible generation of mass in the Abelian Higgs model based not on the Higgs mechanism but on the requirement of integrability of the equations of motion and of absence of essential singularities. In any case, the existence of zeroes $H(x) = -m_{em}/|e|$ is compatible with the existence, in the framework of monopoles, of static finite-energy solutions with a non-trivial behaviour at space infinity [19] in the case of two dimensions (Nielsen-Olesen vortices [22]), whose approximate existence in 3+1 dimensions is welcome for the theory of superconductivity.

In a type I superconductor the ratio $\kappa = \tilde{\lambda}/\xi = m_H/\sqrt{2}m_{em}$ of the magnetic field penetration depth $\tilde{\lambda} = 1/\sqrt{2}m_{em}$ and of the coherence length $\xi = 1/m_H$ [it gives a scale for the variations of the order parameter $\phi(x)$] satisfies $\sqrt{2}\kappa \leq 1$, which corresponds to $m_H \leq m_{em}$ in the simulation of the Ginzburg-Landau theory with Eq.(1) [4]; in this case there are only the electromagnetic (normal state) and the Higgs (superconducting state) phases and there is a critical value for an external magnetic field at which the order parameter changes discontinously from ϕ_o to zero and the superconductor returns to the normal state (no Meissner effect) with a first-order phase transition.

When $\sqrt{2}\kappa > 1$ or $m_H > m_{em}$, one has type II superconductors, in which a third phase (vortices of magnetic field inside the material in the superconducting state) is present and in which $< \phi(x) >= \phi_o(x)$ [i.e. ϕ_o is spatially varying]. To see the possibility of the occurence of these vortices in the simulation with the Lagrangian density of Eq.(1) with the description involving only Dirac's observables, let us consider the physical Lagrangian density of Eq.(26) in the Higgs phase in absence of fermions, for vanishing electric field $\vec{\pi}(x) = \vec{E}(x) = 0$ and in the static case $\partial^o \vec{A}^{'}(x) = \partial^o H(x) = 0$; moreover, let us suppose to have cylindrical symmetry, $A^{'}_3(\vec{x}) = 0$ and $\partial_3 \vec{A}^{'}(\vec{x}) = 0$ [so that $B_1(\vec{x}) = B_2(\vec{x}) = 0$, $B_3(\vec{x}) = -F_{12}(\vec{x})$] and $\partial_3 H(\vec{x}) = 0$. Let us also take $\phi_o = 1$, $e = \frac{1}{2}$, $\lambda = \frac{1}{8}$, so that $m_{em} = m_H = 1/\sqrt{2}$ [this is the critical value separating type I from type II superconductors]. Then the physical Lagrangian density (26) reduces to [we follow Ref. [23] and use the redefinition $H^{'} = H/\sqrt{2}$]

$$
\begin{aligned}
\mathcal{L}"(x^1, x^2) &= -\frac{1}{2}F^2_{12} - \frac{1}{4}(1 + \frac{1}{\sqrt{2}}H)^2(A^{'2}_1 + A^{'2}_2) - \frac{1}{2}[(\partial_1 H)^2 + (\partial_2 H)^2] - \\
&- \frac{1}{4}H^2(1 + \frac{1}{2\sqrt{2}}H)^2 = \\
&= -\frac{1}{2}F^2_{12} - \frac{1}{4}(1 + H^{'})^2(A^{'2}_1 + A^{'2}_2) - (\partial_1 H^{'})^2 - (\partial_2 H^{'})^2 - \frac{1}{8}[(1 + H^{'})^2 - 1]^2 =
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{2}\{(1+H^{'})^2[\frac{1}{2}(A_1^{'2}+A_2^{'2})+2(\partial_1 ln(1+H^{'}))^2+2(\partial_2 ln(1+H^{'}))^2 - \\
&- \partial_1 A_2^{'}+\partial_2 A_1^{'}]+[F_{12}+\frac{1}{2}((1+H^{'})^2-1)]^2+F_{12}\} = \\
&= -\frac{1}{2}\{(1+H^{'})^2[(\frac{A_1^{'}}{\sqrt{2}}-\sqrt{2}\partial_2 ln(1+H^{'}))^2+(\frac{A_2^{'}}{\sqrt{2}}+\sqrt{2}\partial_1 ln(1+H^{'}))^2]+ \\
&+ [F_{12}+\frac{1}{2}((1+H^{'})^2-1)]^2+F_{12}-\partial_1[(1+H^{'})^2A_2^{'}]+\partial_2[(1+H^{'})^2A_1^{'}]\}. \qquad (42)
\end{aligned}
$$

Therefore, modulo surface terms,the static 2-dimensional action $S^{"} = -\int d^2x\, \mathcal{L}^{"}(x^1,x^2)$ is positive definite except for the term $-\frac{1}{2}\int d^2x\, F_{12}$. We get $-S^{"} = +\frac{1}{2}\int d^2x F_{12}$ [the lower bound of Bogomol'nyi [24]; the conditions for having finite action are $|\phi| = 1+H^{'} \rightarrow_{r\rightarrow\infty} 1$, $e^{ie\theta}D_\mu^{(A)}\phi = \partial_\mu H^{'} - ie(1+H^{'})A_\mu^{'} \rightarrow_{r\rightarrow\infty} 0$] if

$$
\begin{aligned}
&A_1^{'} = 2\partial_2 ln(1+H^{'}), \qquad A_2^{'} = -2\partial_1 ln(1+H^{'}), \qquad \Rightarrow F_{12} = 2\triangle ln(1+H^{'}), \\
&F_{12} = -\frac{1}{2}[(1+H^{'})^2-1], \\
&\Downarrow \\
&\triangle ln(1+H^{'}) = -\frac{1}{4}[(1+H^{'})^2-1]. \qquad (43)
\end{aligned}
$$

This is the form of the equations for the Nielsen-Olesen vortices [22] in terms of Dirac observables when $\lambda = \frac{1}{8}$. For the vortex solution $[|\phi(\vec{x})| \rightarrow_{r\rightarrow\infty} 1,\, |\vec{A}^{'}(\vec{x})| \rightarrow_{r\rightarrow\infty} \frac{1}{er}+\frac{c}{e}\sqrt{\frac{\pi}{2er}}e^{-er}]$, one has $S^{"} = -\pi n$ with $n = \frac{1}{2\pi}\int d^2x F_{12}$, a topological invariant measuring the order of vanishing of $|\phi| = 1+H$ in a discrete set of points $\vec{x}_k$ around which $1+H(\vec{x}) = |\vec{x}-\vec{x}_k|^{n_k}+\dots$ with $n = \sum_k n_k$ (see Ref. [23]). One recovers the electromagnetic phase $[m_{em}+|e|H = 0]$ in these points, where the phase θ [satisfying $\theta(\vec{x}) = \theta(|\vec{x}|,\varphi) = \theta(|\vec{x}|,\varphi+2\pi n)$] and the θ-dependent gauge transformation are singular.

V. THE REFORMULATION ON SPACELIKE HYPERSURFACES

Both the phases are not described in a Lorentz-covariant way. Te remedy it, let us reformulate the Lagrangian density of Eq.(1), in absence of fermions for the sake of simplicity, on a family of spacelike hypersurfaces foliating the Minkowski spacetime, along the lines of

Refs. [1,8,25]. We skip all the details of the construction, which is fully explained in Ref. [8], and only sketch the starting point and the final results.

If $z^\mu(\tau,\vec{\sigma})$ are the Minkowski coordinates of the points of the spacelike hypersurface (each leaf of the foliation is identified by the value of a scalar parameter τ), whose curvilinear coordinates are $\vec{\sigma}$, $g_{AB}(\tau,\vec{\sigma}) = \frac{\partial z^\mu(\tau,\vec{\sigma})}{\partial\sigma^A}\eta_{\mu\nu}\frac{\partial z^\nu(\tau,\vec{\sigma})}{\partial\sigma^B} = z^\mu_A(\tau,\vec{\sigma})\eta_{\mu\nu}z^\nu_B(\tau,\vec{\sigma})$ $[A=\tau,\check{r};\ \sigma^\tau=\tau]$ the metric tensor induced on the hypersurface and $A_A(\tau,\vec{\sigma}) = z^\mu_A(\tau,\vec{\sigma})A_\mu(z(\tau,\vec{\sigma}))$ and $\tilde{\phi}(\tau,\vec{\sigma}) = \phi(z(\tau,\vec{\sigma}))$ the electromagnetic potential and the Higgs field respectively, Eq.(1) is replaced by

$$
\begin{aligned}
\tilde{\mathcal{L}}(\tau,\vec{\sigma}) = &\frac{1}{2}\sqrt{g(\tau,\vec{\sigma})}\\
&\{g^{\tau\tau}[D^{(A)}_\tau\tilde{\phi}]^* D^{(A)}_\tau\tilde{\phi} + g^{\tau\check{r}}([D^{(A)}_\tau\tilde{\phi}]^* D^{(A)}_{\check{r}}\tilde{\phi} + [D^{(A)}_{\check{r}}\tilde{\phi}]^* D^{(A)}_\tau\tilde{\phi}) + g^{\check{r}\check{s}}[D^{(A)}_{\check{r}}\tilde{\phi}]^* D^{(A)}_{\check{s}}\tilde{\phi} -\\
&- V(\tilde{\phi}) - \frac{1}{2}g^{AC}g^{BD}F_{AB}F_{CD}\}(\tau,\vec{\sigma}).
\end{aligned}
\tag{44}
$$

The canonical momenta are $\rho_\mu(\tau,\vec{\sigma}) = \partial\tilde{\mathcal{L}}(\tau,\vec{\sigma})/\partial z^\mu_\tau(\tau,\vec{\sigma})$, $\pi^\tau(\tau,\vec{\sigma}) = \partial\tilde{\mathcal{L}}/\partial\partial_\tau A_\tau(\tau,\vec{\sigma}) = 0$, $\pi^{\check{r}}(\tau,\vec{\sigma}) = \partial\tilde{\mathcal{L}}(\tau,\vec{\sigma})/\partial\partial_\tau A_{\check{r}}(\tau,\vec{\sigma})$, $\tilde{\pi}_\phi(\tau,\vec{\sigma}) = \partial\tilde{\mathcal{L}}(\tau,\vec{\sigma})/\partial\partial_\tau\tilde{\phi}(\tau,\vec{\sigma})$, $\tilde{\pi}_{\phi^*}(\tau,\vec{\sigma}) = \partial\tilde{\mathcal{L}}(\tau,\vec{\sigma})/\partial\partial_\tau\tilde{\phi}^*(\tau,\vec{\sigma})$: $\pi^\tau(\tau,\vec{\sigma})$ and $\pi^{\check{r}}(\tau,\vec{\sigma})$ are now Lorentz-scalars. We find five primary constraints

$$
\begin{aligned}
&\mathcal{H}_\mu(\tau,\vec{\sigma}) = \rho_\mu(\tau,\vec{\sigma}) - l_\mu(\tau,\vec{\sigma})\Theta^{\tau\tau}(\tau,\vec{\sigma}) - z_{\check{r}\mu}(\tau,\vec{\sigma})\Theta^{\tau\check{r}}(\tau,\vec{\sigma}) \approx 0\\
&\pi^\tau(\tau,\vec{\sigma}) \approx 0,
\end{aligned}
\tag{45}
$$

where $l_\mu(\tau,\vec{\sigma})$ is the normal to the hypersurface, built only in terms of its tangent vectors $z^\mu_{\check{r}}(\tau,\vec{\sigma})$. Since the canonical Hamiltonian vanishes, the Dirac Hamiltonian, combination of the primary constraints, implies only the secondary Lorentz-scalar constraint (Gauss law) $\Gamma(\tau,\vec{\sigma}) = -\partial^{\check{r}}\pi^{\check{r}}(\tau,\vec{\sigma}) - ie(\tilde{\pi}_\phi\tilde{\phi} - \tilde{\pi}_{\phi^*}\tilde{\phi}^*)(\tau,\vec{\sigma}) \approx 0$. All the constraints are first class.

For the main stratum of field configurations with total timelike momentum, $P^2>0$, we can reduce the theory to the Wigner hyperplanes orthogonal to P^μ. After the reduction, $\vec{A}(\tau,\vec{\sigma})$ and $\vec{\pi}(\tau,\vec{\sigma})$ become Wigner spin-1 3-vectors and the pairs $z^\mu(\tau,\vec{\sigma}), \rho_\mu(\tau,\vec{\sigma})$ are reduced to a point $\tilde{x}^\mu_s(\tau), p^\mu_s \approx P^\mu$, which are the canonical coordinates of the center of mass of the configuration of fields [$\tilde{x}^\mu_s$ is not a four-vector] and, if we denote $\epsilon_s = \pm\sqrt{p^2_s}$ the invariant mass of the system, we are left only with the constraints

$$\begin{aligned}
&\mathcal{H}(\tau) = \epsilon_s - \int d^3\sigma \{\frac{1}{2}[\vec{\pi}^2(\tau,\vec{\sigma}) + \vec{B}^2(\tau,\vec{\sigma})] + \tilde{\pi}_{\phi^*}(\tau,\vec{\sigma})\tilde{\pi}_\phi(\tau,\vec{\sigma}) + \\
&\quad + [(\vec{\partial} + ie\vec{A}(\tau,\vec{\sigma}))\tilde{\phi}^*(\tau,\vec{\sigma})] \cdot (\vec{\partial} - ie\vec{A}(\tau,\vec{\sigma}))\tilde{\phi}(\tau,\vec{\sigma}) + \lambda(\tilde{\phi}^*(\tau,\vec{\sigma})\tilde{\phi}(\tau,\vec{\sigma}) - \phi^2_o)^2\} \approx 0 \\
&\vec{\mathcal{H}}(\tau) = \int d^3\sigma \{\vec{\pi}(\tau,\vec{\sigma}) \times \vec{B}(\tau,\vec{\sigma}) + \\
&\quad + \tilde{\pi}_\phi(\tau,\vec{\sigma})(\vec{\partial} - ie\vec{A}(\tau,\vec{\sigma}))\tilde{\phi}(\tau,\vec{\sigma}) + \tilde{\pi}_{\phi^*}(\tau,\vec{\sigma})(\vec{\partial} + ie\vec{A}(\tau,\vec{\sigma}))\tilde{\phi}^*(\tau,\vec{\sigma})\} \approx 0 \\
&\pi^\tau(\tau,\vec{\sigma}) \approx 0 \\
&\Gamma(\tau,\vec{\sigma}) \approx 0.
\end{aligned} \tag{46}$$

The constraints $\vec{\mathcal{H}} \approx 0$ say that the hyperplane defines an intrinsic rest frame for the system of fields; its gauge-fixings would force the center of mass of the system defined inside the hyperplane to coincide with x^μ_s [the center of mass defined from outside the hyperplane, taking into account its embedding in Minkowski spacetime]. This is the covariant rest-frame instant form of the dynamics [8].

We see that the reduction to either the electromagnetic or the Higgs phase may be done as before, but now in a Lorentz invariant way.

In Eq.(44) the configuration variables are $z^\mu(\tau,\vec{\sigma})$, $A_A(\tau,\vec{\sigma})$ and $\tilde{\phi}(\tau,\vec{\sigma})$. As shown in Appendix C of Ref. [8], instead of the gauge potentials $A_\tau(\tau,\vec{\sigma})$, $A_{\check{r}}(\tau,\vec{\sigma})$ one can use $A_l(\tau,\vec{\sigma})$, $A_{\check{r}}(\tau,\vec{\sigma})$ with $A_\mu(z(\tau,\vec{\sigma})) = z^A_\mu(\tau,\vec{\sigma})A_A(\tau,\vec{\sigma}) = l_\mu(\tau,\vec{\sigma})A_l(\tau,\vec{\sigma}) + z^{\check{r}}_\mu(\tau,\vec{\sigma})A_{\check{r}}(\tau,\vec{\sigma})$ [here $Z^A_\mu(\tau,\vec{\sigma})$ are the vierbeins inverse of $z^\mu_A(\tau,\vec{\sigma})$]. In this way Eq.(44) is replaced by

$$\begin{aligned}
\tilde{\mathcal{L}}(\tau,\vec{\sigma}) &= \frac{1}{2}\frac{\sqrt{\gamma(\tau,\vec{\sigma})}}{N(\tau,\vec{\sigma})}\{[D^{(A)}_\tau\tilde{\phi}]^* D^{(A)}_\tau\tilde{\phi} - N^{\check{r}}([D^{(A)}_\tau\tilde{\phi}]^* D^{(A)}_{\check{r}}\tilde{\phi} + [D^{(A)}_{\check{r}}\tilde{\phi}]^* D^{(A)}_\tau\tilde{\phi}) + \\
&+ (N^2\gamma^{\check{r}\check{s}} + N^{\check{r}}N^{\check{s}})[D^{(A)}_{\check{r}}\tilde{\phi}]^* D^{(A)}_{\check{s}}\tilde{\phi} - N(\tau,\vec{\sigma})V(\tilde{\phi}) - \\
&- \sqrt{\gamma(\tau,\vec{\sigma})}[\frac{1}{2N}\gamma^{\check{r}\check{s}}(\partial_\tau A_{\check{r}} - \mathcal{L}_{\vec{N}}A_{\check{r}} - \partial_{\check{r}}(NA_l))(\partial_\tau A_{\check{s}} - \mathcal{L}_{\vec{N}}A_{\check{s}} - \partial_{\check{s}}(NA_l)) + \\
&+ \frac{N}{4}\gamma^{\check{r}\check{s}}\gamma^{\check{u}\check{v}}F_{\check{r}\check{u}}F_{\check{s}\check{v}}](\tau,\vec{\sigma})\},
\end{aligned} \tag{47}$$

where $N(\tau,\vec{\sigma}) = \sqrt{g(\tau,\vec{\sigma})/\gamma(\tau,\vec{\sigma})}$ [$\gamma = -det\, |\, g_{\check{r}\check{s}}\, |$], $N^{\check{r}}(\tau,\vec{\sigma}) = g_{\tau\check{s}}(\tau,\vec{\sigma})\gamma^{\check{s}\check{r}}(\tau,\vec{\sigma})$ [$\gamma^{\check{r}\check{u}}g_{\check{u}\check{s}} = \delta^{\check{r}}_{\check{s}}$], are the lapse and shift functions; one has $A_\tau(\tau,\vec{\sigma}) = N(\tau,\vec{\sigma})A_l(\tau,\vec{\sigma}) + N^{\check{r}}(\tau,\vec{\sigma})A_{\check{r}}(\tau,\vec{\sigma})$

and $z^\mu_\tau(\tau,\vec{\sigma}) = N(\tau,\vec{\sigma})l^\mu(\tau,\vec{\sigma}) + N^{\check{r}}(\tau,\vec{\sigma})z^\mu_{\check{r}}(\tau,\vec{\sigma})$. Eq.(47) leads again to Eqs.(46) [with $\pi^l(\tau,\vec{\sigma}) = N(\tau,\vec{\sigma})\pi^\tau(\tau,\vec{\sigma}) \approx 0$], since both $A_\tau(\tau,\vec{\sigma})$ and $A_l(\tau,\vec{\sigma})$ are gauge variables.

Instead the reformulation on spacelike hypersurfaces of the standard massive vector field is

$$\begin{aligned}
\tilde{\mathcal{L}}'(\tau,\vec{\sigma}) &= \sqrt{g(\tau,\vec{\sigma})}\{-\frac{1}{4}g^{AC}g^{BD}F_{AB}F_{CD} + \frac{1}{2}M^2g^{AB}A_AA_B\}(\tau,\vec{\sigma}) = \\
&= -\sqrt{\gamma(\tau,\vec{\sigma})}\{\frac{1}{2N}\gamma^{\check{r}\check{s}}(\partial_\tau A_{\check{r}} - \mathcal{L}_{\vec{N}}A_{\check{r}} - \partial_{\check{r}}(NA_l))(\partial_\tau A_{\check{s}} - \mathcal{L}_{\vec{N}}A_{\check{s}} - \partial_{\check{s}}(NA_l)) + \\
&+ \frac{N}{4}\gamma^{\check{r}\check{s}}\gamma^{\check{u}\check{v}}F_{\check{r}\check{u}}F_{\check{s}\check{v}}](\tau,\vec{\sigma})\} + \\
&+ \frac{1}{2}M^2\sqrt{\gamma(\tau,\vec{\sigma})}N(\tau,\vec{\sigma})\{A_l^2(\tau,\vec{\sigma}) + \gamma^{\check{r}\check{s}}(\tau,\vec{\sigma})A_{\check{r}}(\tau,\vec{\sigma})A_{\check{s}}(\tau,\vec{\sigma})\}.
\end{aligned} \tag{48}$$

The final Lorentz-invariant constraints for $P^2 > 0$ on the hyperplane orthogonal to the total momentum, after the elination of A_l, π^l, are

$$\begin{aligned}
\mathcal{H}(\tau) &= \epsilon_s - \int d^3\sigma\{\frac{1}{2}[\vec{\pi}^2(\tau,\vec{\sigma}) + \vec{B}^2(\tau,\vec{\sigma})] + \frac{1}{2}M^2\vec{A}^2(\tau,\vec{\sigma}) + \frac{(\vec{\partial}\cdot\vec{\pi}(\tau,\vec{\sigma}))^2}{2M^2}\} \approx 0 \\
\vec{\mathcal{H}}(\tau) &= \int d^3\sigma\, \vec{\pi}(\tau,\vec{\sigma}) \times \vec{B}(\tau,\vec{\sigma}) \approx 0.
\end{aligned} \tag{49}$$

Now, on spacelike hypersurfaces there is the possibility to define in a covariant way the Lagrangian density (37), which is replaced by

$$\begin{aligned}
\tilde{\mathcal{L}}"(\tau,\vec{\sigma}) &= -\sqrt{\gamma(\tau,\vec{\sigma})}\{\frac{1}{2N}\gamma^{\check{r}\check{s}}(\partial_\tau A_{\check{r}} - \mathcal{L}_{\vec{N}}A_{\check{r}} - \partial_{\check{r}}(NA_l))(\partial_\tau A_{\check{s}} - \mathcal{L}_{\vec{N}}A_{\check{s}} - \partial_{\check{s}}(NA_l)) + \\
&+ \frac{N}{4}\gamma^{\check{r}\check{s}}\gamma^{\check{u}\check{v}}F_{\check{r}\check{u}}F_{\check{s}\check{v}}](\tau,\vec{\sigma})\} + \\
&+ \frac{1}{2}\sqrt{\gamma(\tau,\vec{\sigma})}N(\tau,\vec{\sigma})\{M^2A_l^2 - \gamma^{\check{r}\check{s}}\partial_{\check{r}}A_l\partial_{\check{s}}A_l + M^2\gamma^{\check{r}\check{s}}A_{\check{r}}A_{\check{s}}\}(\tau,\vec{\sigma}).
\end{aligned} \tag{50}$$

The final reduced Lorentz-invariant constraint on the hyperplane orthogonal to the momentum are

$$\begin{aligned}
\mathcal{H}(\tau) &= \epsilon_s - \int d^3\sigma\{\frac{1}{2}[\vec{\pi}^2(\tau,\vec{\sigma}) + \vec{B}^2(\tau,\vec{\sigma})] + \frac{1}{2}M^2\vec{A}^2(\tau,\vec{\sigma}) + \\
&+ \frac{1}{2}\vec{\partial}\cdot\vec{\pi}(\tau,\vec{\sigma})\frac{1}{\triangle + M^2}\vec{\partial}\cdot\vec{\pi}(\tau,\vec{\sigma})\} \approx 0 \\
\vec{\mathcal{H}}(\tau) &= \int d^3\sigma\, \vec{\pi}(\tau,\vec{\sigma}) \times \vec{B}(\tau,\vec{\sigma}) \approx 0.
\end{aligned} \tag{51}$$

VI. COMMENTS

Let us make some final comments:

i) The same ambiguity in solving the Gauss law constraint, which originates the two phases, is consistently present in the covariant R-gauge-fixing [26]

$$\partial^\mu A_\mu(x) + \frac{1}{\xi}\theta(x) \approx 0 \tag{52}$$

used in the covariant-gauge approach to renormalization (to remedy the nonrenormalizability of the unitary gauge) and in the evaluation of radiative corrections with the associated Feynman rules (see for instance Ref. [12]). Therefore, in these procedures one is mixing the two phases except in the final $\xi \rightarrow \infty$ limit to reach the unitary gauge. Moreover, in the perturbative calculations one cannot see the nonanalyticity in the coupling constant e (or in $m_{em} + |e|H(x)$) of the electric phenomena in the Higgs phase.

ii) In Eq.(26), the residual real scalar Higgs field $H(x)$ is actually coupled only to $|e|/m_{em}$; now this quantity appear in the mass term of the vector gauge field and one is tempted to say that $H(x)$ is charged but not minimally coupled to the electromagnetic field. To understand what is going on we must study the conserved charges associated to the Gauss law in both phases. This is not trivial due to the fact that in the broken gauge symmetry Higgs phase the electric and magnetic fields decay at space infinity with Yukawa tails due to the mass m_{em}. Therefore, the Gauss theorem breaks down: the electric charge in the Higgs phase is a Noether constant of motion (first Noether theorem) but one cannot measure it by means of the electric flux at space infinity (as in the case of exact, not broken, local gauge symmetry; second Noether theorem [18]). This fact may be taken as a gauge-invariant signal of gauge symmetry breaking, rather than the non-gauge-invariant quantum statement $< \phi >= \phi_o$ (see Ref. [27] for a criticism of this criterion).

The Euler-Lagrange equations associated with the Lagrangian density of Eq.(1) are

$$\begin{aligned} L^\mu &= \frac{\partial \mathcal{L}}{\partial A_\mu} - \partial_\nu \frac{\partial \mathcal{L}}{\partial \partial_\nu A_\mu} = \partial_\nu F^{\nu\mu} + eJ^\mu \overset{\circ}{=} 0 \\ J^{\ \mu} &= \bar{\psi}\gamma^\mu\psi + i\phi^*[(\partial^\mu - ieA^\mu) - (\overleftarrow{\partial^\mu + ie}A^\mu)]\phi \end{aligned}$$

$$
\begin{aligned}
L_\psi &= \frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi} = -\bar{\psi}[(i\overleftarrow{\partial_\mu - eA_\mu})\, \gamma^\mu + m] \stackrel{\circ}{=} 0 \\
L_{\bar{\psi}} &= \frac{\partial \mathcal{L}}{\partial \bar{\psi}} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \bar{\psi}} = [\gamma^\mu (i\partial_\mu + eA_\mu) - m]\psi \stackrel{\circ}{=} 0 \\
L_\phi &= \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} = -[D^{(A)\mu} D^{(A)}_\mu \phi]^* - \frac{\partial V(\phi)}{\partial \phi} \stackrel{\circ}{=} 0 \\
L_{\phi^*} &= \frac{\partial \mathcal{L}}{\partial \phi^*} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^*} = -D^{(A)\mu} D^{(A)}_\mu \phi - \frac{\partial V(\phi)}{\partial \phi^*} \stackrel{\circ}{=} 0.
\end{aligned}
\tag{53}
$$

Let us note that in presence of external electromagnetic fields [so that $A_\mu \mapsto A_\mu + A_{ext,\mu}$] the Euler-Lagrange equations of the Higgs field are solved by requiring [28]

$$
\begin{aligned}
& D^{(A+A_{ext})}_\mu \phi \stackrel{\circ}{=} 0 \\
& \frac{\partial V(\phi)}{\partial \phi} \stackrel{\circ}{=} 0.
\end{aligned}
\tag{54}
$$

While the second equation has the two solutions $\phi = 0$ and $\phi = \phi_o$, from the first equation we get $0 \stackrel{\circ}{=} [D^{(A+A_{ext})}_\mu, D^{(A+A_{ext})}_\nu]\phi = -ie[F_{\mu\nu} + F_{ext,\mu\nu}]\phi$. Therefore, we get either $\phi = 0$ and $F + F_{ext} \neq 0$ (the electromagnetic phase) or $\phi = \phi_o$ and the Meissner effect $F + F_{ext} = 0$ (the Higgs phase).

The gauge invariance of $\mathcal{L}(x)$ under the infinitesimal gauge transformations $\delta A_\mu = -\frac{1}{e}\partial_\mu \alpha$, $\delta \psi = -i\alpha\psi$, $\delta \bar{\psi} = i\bar{\psi}\alpha$, $\delta\phi = -i\alpha\phi$, $\delta\phi^* = i\alpha\phi^*$, produces the Noether identities

$$
\begin{aligned}
0 \equiv \delta\mathcal{L} &= \frac{\partial \mathcal{L}}{\partial A_\mu}\delta A_\mu + \frac{\partial \mathcal{L}}{\partial \partial_\nu A_\mu}\delta \partial_\nu A_\mu + \delta\bar{\psi}\frac{\partial \mathcal{L}}{\partial \bar{\psi}} + \delta\partial_\mu \bar{\psi}\frac{\partial \mathcal{L}}{\partial \partial_\mu \bar{\psi}} + \\
&+ \delta\psi \frac{\partial \mathcal{L}}{\partial \psi} + \delta\partial_\mu \psi \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi} + \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\delta\partial_\mu \phi + \\
&+ \frac{\partial \mathcal{L}}{\partial \phi^*}\delta\phi^* + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^*}\delta\partial_\mu \phi^* = \\
&= L^\mu \delta A_\mu + \delta\bar{\psi} L_{\bar{\psi}} - L_\psi \delta\psi + \delta\phi^* L_{\phi^*} + L_\phi \delta\phi + \partial_\mu G^\mu
\end{aligned}
$$

$$
\begin{aligned}
G^\mu &= \alpha G^\mu_1 + \partial_\nu \alpha G^{\mu\nu}_o = \\
&= -F^{\mu\nu}\delta A_\nu - \frac{i}{2}[\delta\bar{\psi}\gamma^\mu\psi - \bar{\psi}\gamma^\mu\delta\psi] + [D^{(A)\mu}\phi]^*\delta\phi + \delta\phi^* D^{(A)\mu}\phi \\
&\Downarrow
\end{aligned}
$$

$$
\begin{aligned}
G_o^{\mu\nu} &= \frac{1}{e} F^{\mu\nu}\\
G_1^{\mu} &= \bar{\psi}\gamma^\mu\psi + i(\phi^* D^{(A)\mu}\phi - [D^{(A)\mu}\phi]^*\phi) = J^\mu
\end{aligned}
$$

$$
\begin{aligned}
\partial_\mu G^\mu &= \partial_\mu\partial_\nu\alpha G_o^{\mu\nu} + \partial_\mu\alpha[\partial_\nu G_o^{\nu\mu} + G_1^\mu] + \alpha\partial_\mu G_1^\mu \equiv\\
&\equiv -L^\mu\delta A_\mu + L_\psi\delta\psi - \delta\bar{\psi}L_{\bar{\psi}} - L_\phi\delta\phi - \delta\phi^* L_{\phi^*} \overset{\circ}{=} 0.
\end{aligned}
\tag{55}
$$

The last line implies the Noether identities [$(\mu\nu)$ and $[\mu\nu]$ mean symmetrization and antisymmetrization respectively]

$$
\begin{aligned}
&G_o^{(\mu\nu)} \equiv 0\\
&\partial_\nu G_o^{\nu\mu} \equiv -G_1^\mu + \frac{1}{e}L^\mu = \frac{1}{e}L^\mu - \bar{\psi}\gamma^\mu\psi - i(\phi^* D^{(A)\mu}\phi - [D^{(A)\mu}\phi]^*\phi)\\
&\partial_\mu G_1^\mu \equiv -i(L_\psi\psi + \bar{\psi}L_{\bar{\psi}}) + i(L_\phi\phi - \phi^* L_{\phi^*}) \overset{\circ}{=} 0
\end{aligned}
\tag{56}
$$

and, from the last two lines of these equations, the contracted Bianchi identities

$$
\partial_\mu L^\mu + ie(L_\psi\psi + \bar{\psi}L_{\bar{\psi}} + \phi^* L_{\phi^*} - L_\phi\phi) \equiv 0.
\tag{57}
$$

The following subset of Noether identities reproduces the Hamiltonian constraints

$$
\begin{aligned}
&\pi^o = -eG_o^{(oo)} \equiv 0\\
&0 \equiv \partial^o\pi^o \equiv \partial^k\pi^k - eJ^o - L^o = -\Gamma - L^o \overset{\circ}{=} -\Gamma.
\end{aligned}
\tag{58}
$$

The strong improper conservation law [18] $\partial_\mu V^\mu \equiv 0$, implied by Eqs.(56), identifies the strong improper conserved current (strong continuity equation)

$$
\begin{aligned}
V^\mu &= -\partial_\nu G_o^{\nu\mu} = -\frac{1}{e}\partial_\nu F^{\nu\mu} = \partial_\nu U^{[\mu\nu]} \overset{\circ}{=} J^\mu =\\
&= \bar{\psi}\gamma^\mu\psi + i(\phi^* D^{(A)\mu}\phi - [D^{(A)\mu}\phi]^*\phi) = G_1^\mu = j_F^\mu + j_{KG}^\mu,
\end{aligned}
\tag{59}
$$

with the superpotential $U^{[\mu\nu]} = \frac{1}{e}F^{\mu\nu}$. In the last line, $j_f^\mu = \bar{\psi}\gamma^\mu\psi$ and $j_{KG}^\mu = i(\phi^* D^{(A)\mu}\phi - [D^{(A)\mu}\phi]^*\phi)$ are the charge currents of the fermion field and of the complex Klein-Gordon Higgs fields respectively.

The associated weak improper conservation law is $\partial_\mu G_1^\mu \circeq 0$ [it is obtained by using the second line of Eqs.(56)]. If Q is the weak improper conserved Noether charge and $Q^{(V)}$ the strong improper conserved one, we get [its meaning is equivalent to $\int d^3x \Gamma(\vec{x}, x^o) \circeq 0$]

$$
\begin{aligned}
Q &= \int d^3x G_1^o(\vec{x}, x^o) = \int d^3x J^o(\vec{x}, x^o) = \\
&= \int d^3x [\bar{\psi}(\vec{x}, x^o)\gamma^o\psi(\vec{x}, x^o) - i(\pi_\phi(\vec{x}, x^o)\phi(\vec{x}, x^o) - \phi^*(\vec{x}, x^o)\pi_{\phi^*}(\vec{x}, x^o))] = \\
&= \int d^3x [\psi^\dagger(\vec{x}, x^o)\psi(\vec{x}, x^o) - \pi_\theta(\vec{x}, x^o)] = Q_F + Q_\theta \circeq \\
\circeq Q^{(V)} &= \int d^3x V^o(\vec{x}, x^o) = \int d^3x \partial^k F^{ko}(\vec{x}, x^o) = \int d^3x \vec{\partial} \cdot \vec{\pi}(\vec{x}, x^o) = \int d\vec{\Sigma} \cdot \vec{E}(\vec{x}, x^o),
\end{aligned}
\tag{60}
$$

where Q_F and Q_θ are the electric charges (in units of e) of the fermion fields and of the complex Higgs field.

In the electromagnetic phase, $Q \circeq Q^{(V)}$ is the Gauss theorem associated with the long-range electromagnetic interaction: the flux at space infinity of the electric field is equal to the total electric charge of the fermions and of the charged complex Higgs fields, dressed with their Coulomb clouds [$Q = \int d^3x [\check{\bar{\psi}}^\dagger \check{\psi} - i(\check{\pi}_\phi \check{\phi} - \check{\phi}^* \check{\pi}_{\phi^*})](\vec{x}, x^o)$], with the additional information that the Higgs electric charge is carried by the phase $\theta(x)$.

On the contrary, in the broken symmetry Higgs phase we get $Q^{(V)} = 0$ when Eq.(60) is integrated over all the 3-space, because the electric field decays exponentially at space infinity due to the generated electromagnetic mass m_{em} (short-range interaction), so that the Gauss theorem breaks down in presence of spontaneous symmetry breaking through the Higgs mechanism. The residual Higgs field $H(x)$ turns out to be neutral, being instead coupled to the ratio $|e|/m_{em}$ and the electromagnetic mass is replaced by the effective mass $m_{em}(1 + \frac{|e|}{m_{em}} H)$ [one could also say that the last term of Eq.(25) describes an effective mass $m_H(1 + \frac{|e|}{2m_{em}} H)$ for H itself]. When we integrate over all 3-space, Eq.(60) can be written as

$$
\begin{aligned}
&Q = \hat{Q}_F + Q_\theta \circeq 0 \\
&Q_\theta \circeq - \hat{Q}_F = - \int d^3x [\hat{\bar{\psi}}^\dagger \hat{\psi}](\vec{x}, x^o),
\end{aligned}
\tag{61}
$$

and says that the charge Q_θ of the nonlinearly interacting would-be massless Goldstone boson $\theta(x)$ [which does not appear among the Dirac's observables, being eaten by the vector field, and which has the quantum numbers of the broken generator of U(1) at the quantum level (see Ref. [9] for the infrared singularity associated with this unphysical massless would-be Goldstone boson)] is opposite to the fermionic electric charge $\hat{Q}_F$, which, by itself, is an ordinary conserved Noether charge due to the invariance of the physical Lagrangian density of Eq.(26) under global phase transformations of the fermion fields: $\psi \mapsto e^{-i\alpha}\psi$ implies

$$\frac{d}{dx^o}\hat{Q}_F \overset{\circ}{=} 0. \tag{62}$$

Eq.(61) is consistent with the fact that each fermion field is dressed with a Higgs cloud of θ-field which screens the fermion electric charge if looked from space infinity in the way of Eq.(60) in the Higgs phase of the original theory before the reduction to Dirac's observables; the absence of Gauss' theorem is also evident in the self-energy term in the physical Hamiltonian of Eq.(28). When Eq.(60) is integrated over a finite domain $\mathcal{V}$, we have $Q_\theta \overset{\circ}{=} -\hat{Q}_F + \int_{\mathcal{V}} d\vec{\Sigma} \cdot \vec{E}(\vec{x}, x^o)$.

iii) As a last remark, we note that the Lagrangian densities associated to Dirac's observables are in general nonlocal and nonpolynomial, so that the standard regularization and renormalization prescriptions do not hold. In Refs. [1,8] it is shown that for every extended relativistic system (particles, strings, field configurations) in an irreducible timelike poincaré representation one can define a classical unit of length $\rho = \sqrt{-W^2}/P^2$ in terms of the Poincaré Casimirs from the discussion of the center-of-mass problem (ρ is a measure of the domain in 3-space defined by the noncovariance of the center-of-mass coordinate $\tilde{x}^\mu_s$). This can, we hope, be the basis for a ultraviolet cutoff for the quantization of theories formulated on spacelike hypersurfaces (classical background of the Tomonaga-Schwinger approach).

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